THE TWENTIETH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE annual meeting of the Society was this year held in New York City, on Tuesday and Wednesday, December 30–31, extending through the usual morning and afternoon sessions on each day. The attendance included the following eighty-one members:

Mr. E. S. Allen, Professor C. S. Atchison, Dr. Ida Barney, Mr. R. D. Beetle, Professor G. D. Birkhoff, Professor C. L. Bouton, Professor Pierre Boutroux, Professor Joseph Bowden, Dr. C. E. Brooks, Professor E. W. Brown, Mr. R. W. Burgess, Professor B. H. Camp, Dr. Emily Coddington, Dr. A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Dr. C. F. Craig, Dr. W. H. Cramblet, Professor G. H. Cresse, Professor L. P. Eisenhart, Professor G. C. Evans, Professor F. C. Ferry, Professor J. C. Fields, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor T. M. Focke, Dr. W. C. Graustein, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor J. G. Hardy, Professor H. E. Hawkes, Professor L. A. Howland, Professor L. S. Hulburt, Professor E. V. Huntington, Dr. W. A. Hurwitz, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Professor W. R. Longley, Professor C. R. MacInnes, Professor James Maclay, Professor H. W. March, Mr. B. E. Mitchell, Dr. H. H. Mitchell, Dr. F. M. Morgan, Professor Frank Morley, Professor Richard Morris, Mr. G. W. Mullins, Professor G. D. Olds, Professor W. F. Osgood, Dr. F. W. Owens, Dr. Anna Pell, Professor James Pierpont, Professor A. D. Pitcher, Professor Arthur Ranum, Dr. H. W. Reddick, Professor R. G. D. Richardson, Mr. L. B. Robinson, Mrs. E. D. Roe, Jr., Professor W. P. Russell, Professor L. P. Siceloff, Dr. Clara E. Smith, Mr. F. H. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. D. Thompson, Professor J. N. Van der Vries, Mr. H. S. Vandiver, Mr. J. N. Vedder, Mr. H. E. Webb, Professor H. S. White, Dr. E. E. Whitford, Professor E. B. Wilson, Professor W. A. Wilson, Professor Ruth G. Wood.

This meeting was especially marked as the occasion of the delivery by Professor H. B. Fine of his presidential address, on "An unpublished theorem of Kronecker respecting numerical equations."

Ex-President W. F. Osgood occupied the chair at the opening session, yielding it later to ex-President Fine. The Council announced the election of the following persons to membership in the Society: Professor Pierre Boutroux, Princeton University; Mr. E. H. Clarke, Purdue University; Dr. W. H. Cramblet, University of Rochester; Mr. H. J. Ettlinger, University of Texas; Professor W. S. Franklin, Lehigh University; Mr. Haig Galajikian, Princeton University; Professor W. W. Hart, University of Wisconsin; Mr. Barnem Libby, University of Michigan; Mr. G. W. Mullins, Columbia University; Mr. J. A. Northcott, Columbia University; Dr. Mildred L. Sanderson, University of Wisconsin; Mr. J. M. Stetson, Princeton University. Nine applications for membership in the Society were received.

The total membership of the Society is now 703, including 69 life members. The total attendance of members at all meetings of the past year was 418; the number of papers read was 240. The number of members attending at least one meeting during the year was 237. At the annual election 203 votes were cast. The Library now contains 4,902 volumes, excluding unbound dissertations. The Treasurer's report shows a balance of $9,153.58, including the life membership fund of $4,800.82. Sales of the Society's publications during the year amounted to $2,111.45.

A proposed amendment to the Constitution was submitted by the Council, making the secretary of the Chicago Section an ex-officio member of the Council. Final action on this amendment will be taken at the February meeting.

The annual dinner on Tuesday evening, attended by forty-seven members, was as always a most enjoyable occasion.

At the annual election, which closed on Wednesday morning, the following officers and other members of the council were chosen:

Vice-Presidents,	Professor L. P. EISENHART, Professor E. J. WILCZYNSKI.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1916,

Professor C. N. HASKINS, Professor E. V. HUNTINGTON,
Professor L. M. HOSKINS, Professor H. L. RIETZ.

The following papers were read at the annual meeting:

(1) Professor L. L. DINES: "Complete existential theory of Sheffer's postulates for Boolean algebras."

(2) Professor ARNOLD EMCH: "Two convergency proofs."

(3) Professor J. L. COOLIDGE: "Congruences and complexes of circles."

(4) Dr. DUNHAM JACKSON: "On the degree of convergence of Sturm-Liouville series."

(5) Professor VIRGIL SNYDER: "Birational transformations of the cubic variety in four-dimensional space."

(6) Miss A. H. TAPPAN: "Plane sextic curves invariant under a group of linear transformations" (preliminary communication).

(7) Professor C. L. BOUTON: "Explicit formulas for the inverse of an analytic transformation in n variables."

(8) Professor EDWARD KASNER: "The classification of conformal transformations."

(9) Mr. L. B. ROBINSON: "Questions of logic arising from the study of systems of partial differential equations" (preliminary report).

(10) Professor PIERRE BOUTROUX: "On a family of rational differential equations of the first order."

(11) Professor H. B. FINE, Presidential Address: "An unpublished theorem of Kronecker respecting numerical equations."

(12) Dr. W. A. HURWITZ: "Note on the Fredholm determinant."

(13) Professor G. D. BIRKHOFF: "The restricted problem of three bodies."

(14) Professor E. V. HUNTINGTON: "On the accuracy of the contracted form of Horner's method."

(15) Professor O. E. GLENN: "On an analogy between formal-modular invariants and the class of algebraical invariants called Booleans."

(16) Professor G. C. EVANS: "Green's functions for linear partial differential expressions of the second order, and Green's theorem."

(17) Professor W. R. LONGLEY: "An existence theorem for a certain differential equation of the nth order."

(18) Dr. W. C. GRAUSTEIN: "The real-congruence of complex points, planes, lines."

(19) Dr. H. W. REDDICK: "Conformal invariants of an orthogonal curve net" (preliminary communication).

(20) Professor W. F. OSGOOD: "Liouville's theorem concerning periodic functions of several variables."

(21) Mr. L. M. KELLS: "Complete characterization of dynamical trajectories in n-space."

(22) Dr. W. H. CRAMBLET: "A classification of discontinuous functions and some allied problems."

(23) Dr. J. K. LAMOND: "Note on the reduction of multiple L-integrals to iterated L-integrals."

(24) Professor L. A. HOWLAND: "Functions of n variables which are functions of r combinations of these variables."

(25) Dr. J. I. TRACEY: "Covariant curves of the plane rational quintic."

(26) Professor A. B. COBLE: "Restricted systems of equations."

(27) Professor L. P. EISENHART: "Transformations of surfaces of Voss."

(28) Mr. H. GALAJIKIAN: "A type of non-linear integral equation."

(29) Dr. T. H. GRONWALL: "On systems of linear total differential equations."

(30) Dr. T. H. GRONWALL: "Extension of Laurent's theorem to several variables."

(31) Dr. T. H. GRONWALL: "On approximation by trigonometric sums."

(32) Mr. R. D. BEETLE: "Cyclic systems of osculating circles of curves on a surface."

(33) Dr. G. M. GREEN: "Canonical systems in projective differential geometry, with special reference to the theory of curved surfaces."

(34) Professor J. H. M. WEDDERBURN: "A type of primitive algebra."

Miss Tappan's paper was communicated to the Society through Professor Snyder. Mr. Kells was introduced by

Professor Kasner. In the absence of the authors the papers of Professor Dines, Professor Emch, Miss Tappan, Professor Glenn, Professor Evans, Mr. Galajikian, and Professor Wedderburn were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the October number of the *Transactions* Dr. Sheffer presented an elegant and concise set of five postulates for Boolean algebras, and proved them mutually consistent and independent. In the present paper, Professor Dines shows that these postulates, though independent in the ordinary sense that no four of them imply a fifth, are not *completely independent* in the sense in which that term has been used by Professor E. H. Moore.* A "complete existential theory" is constructed for the five postulates, and it is found that there exists among the postulates and their negatives one and only one general implicational relation, namely that the negative of the first implies the third, fourth, and fifth.

2. In the study of automorphic functions defined within a fundamental domain formed by two non-intersecting circles in the elliptic case and by two tangent circles in the trigonometric case, it is necessary to prove the convergency of certain fundamental series, as has been done by Schottky recently.† The first is the series Σr_λ of the radii of all circles forming the boundaries of the domains belonging to the cyclic loxodromic group. The second is the series

$$\Sigma(x_\lambda - x_\lambda')$$

for the cyclic parabolic group, in which x_λ and x_λ' result from x and x' by applying to them λ times in succession the original substitution. Professor Emch gives direct proofs for the convergence of both series.

3. A circle in space may be expressed parametrically by equating the pentaspherical coordinates of a point to five

* E. H. Moore: "An introduction to a form of general analysis," New Haven Mathematical Colloquium, Yale University Press, p. 82.

† "Ueber eine Funktionenklasse, die der Gleichung $F\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right) = F(x)$ genügt," *Journal für reine und angewandte Mathematik*, vol. 143, pp. 1–24 (May, 1913). See also the same *Journal*, vol. 101, pp. 231–236 (1887).

quadratic functions of an auxiliary variable. In the first part of Professor Coolidge's paper it is shown how this leads to easy and uniform proofs of all those classic theorems about circles which are invariant for inversion. In the second part of the paper the circle is treated as the envelope of the spheres through it, and by an adaptation of the Kummer methods for line geometry several new types of circle congruence are brought to light and discussed. In the third part the same methods are extended to the study of complexes of circles.

4. Dr. Jackson's paper gives the extension of theorems on the degree of convergence of Fourier's series to more general series arising from a linear homogeneous differential equation of the second order, the differential equation and the boundary conditions being those discussed by Kneser in volume 58 of the *Mathematische Annalen*. Kneser, following Liouville, makes use of relations which, when applied to the present problem, render the extension immediate in the simplest cases, and suggest the method of treatment in less simple ones. If the function to be developed has a continuous kth derivative of limited variation, or a $(k-1)$th derivative satisfying a Lipschitz condition, certain other restrictions being imposed in each case, the remainder after n terms of the series does not exceed a constant multiple of $1/n^k$ or of $(\log n)/n^k$ respectively. It is shown also that a method of summation previously applied to render the Fourier series more rapidly convergent in certain cases is equally applicable to the Sturm-Liouville series. For example, if the function developed satisfies a Lipschitz condition, it is possible to represent it approximately by a linear combination of the first n characteristic functions with a maximum error not exceeding a constant multiple of $1/n$.

5. Various methods have been employed to determine whether the three-dimensional cubic variety V in linear space of four dimensions can be birationally mapped on ordinary space S_3. Professor Snyder here attempts to determine whether continuous groups of birational transformations exist which leave V invariant. All such transformations that are known to exist, while forming a continuous series, do not form a continuous group.

The operations considered are central projections of V into itself from points on it, and the polar transformation, defined

as follows: On V lies a two-dimensional series of straight lines; through each point pass six. Associated with each line is a two-dimensional surface, locus of the pole of the line as to the residual conic in each plane passing through it. To find the image of any point P, pass a plane through P and the line, and join P to the pole of the line. The harmonic conjugate of P as to the conjugate points as to the conic is the image. Both transformations are Cremonian and involutorial, but the product of any two or more is not periodic.

6. In Miss Tappan's paper three types of linear transformations are considered: (*a*) Those having the vertices of a triangle for invariant points; (*b*) the permutation group of the three homogeneous point coordinates; (*c*) those not of types (*a*) and (*b*), including the G_{168} and G_{360}. Only one sextic can have a continuous group of linear transformations. The highest cycle in (*a*) is of order 30; the curve belonging to it is of genus 10. Sixty-three distinct types of curves with groups of form (*a*) exist. For a cycle of order 21 and one of order 12, the corresponding curve must also have transformations of type (*b*). The Cremona and Riemann non-linear birational transformations of sextic curves are being studied, but the results are not yet completed.

7. Consider the special transformation

$$(1) \qquad x_i' = \phi_i(x_1, \cdots, x_n) \equiv x_i + \sum_{k=2}^{\infty} \phi_{ik}(x_1 \cdots, x_n)$$

$$(i = 1, 2, \cdots, n),$$

where the given functions $\phi_i(x)$ are analytic at the origin, so that the series $\Sigma\phi_{ik}$ is a convergent power series, and ϕ_{ik} is a homogeneous polynomial of degree k. For the neighborhood of the origin the inverse of (1) will then have the same form, viz.,

$$(2) \qquad x_i = \phi_i'(x_1', \ldots, x_n') \equiv x_i' + \sum_{k=2}^{\infty} \phi_{ik}'(x_1', \cdots, x_n')$$

$$(i = 1, 2, \cdots, n),$$

where the power series $\Sigma\phi_{ik}'$ are convergent. Professor Bouton's paper gives the explicit formulas for the computation of the ϕ_{ik}' in terms of the ϕ_{ik}. We have

$$(3)\quad \phi_{i2}'(x) \equiv -\phi_{i2}(x), \quad \phi_{ik}'(x) \equiv -\phi_{ik}(x) + \sum_{j=1}^{k-2} Q_j \phi_{i,k-j} \qquad (k = 3, 4, \cdots),$$

where Q_j is a homogeneous linear differentiating operator which, applied to a homogeneous polynomial, raises its degree j units. These operators are found successively from the symbolic identity

$$(4)\quad \sum_{1}^{\infty} Q_j \equiv -\sum_{i=1}^{n} \psi_i' \frac{\partial}{\partial x_i} - \frac{1}{2!}\sum_{i=1}^{n}\sum_{t=1}^{n} \psi_s'\psi_t' \frac{\partial^2}{\partial x_s \partial x_t} - \frac{1}{3!}\sum_{r}\sum_{s}\sum_{t} \psi_r'\psi_s'\psi_t' \frac{\partial^3}{\partial x_r \partial x_s \partial x_t} - \cdots$$

by equating operators of like degree, where $\psi_i' \equiv \sum_{k=2}^{\infty} \phi_{ik}'$. For example,

$$Q_1 = \sum_{i=1}^{n} \phi_{i2} \frac{\partial}{\partial x_i}, \quad Q_2 = \sum_{i=1}^{n} (\phi_{i3} - Q_1\phi_{i2}) \frac{\partial}{\partial x_i} - \frac{1}{2!}\sum_{s}\sum_{t} \phi_{s2}\phi_{t2} \frac{\partial^2}{\partial x_s \partial x_t}.$$

The formula for Q_j is readily written down, and thus (3) gives the inverse of (1).

Finally, it is pointed out that any analytic transformation in the neighborhood of a regular point is projectively equivalent to the form (1), so that we have formulas for the inverse of any such transformation.

An application is given to the case of Cremona transformations.

8. In Professor Kasner's classification two conformal transformations (assumed regular at a given point) are regarded as equivalent when the one can be transformed into the other by a conformal transformation. In previous papers the author has discussed (conformal) invariants of analytic curves and of curvilinear angles. The present problem is to find invariants of a conformal transformation $w = f(z)$. The results are closely related to the previous invariants, though the geometric interpretations are quite distinct. The only obvious invariant is w', the value of the first derivative of $f(z)$ at the given point. The simplest example of a higher invariant arises for the class of transformations characterized by $w' = 1$, $w'' \neq 0$, the result being w'''/w''^2, of the third

order. If the transformation has a singularity at the given point, that is, if $w' = 0$, then it can be reduced to the normal form $w = z^k$ $(k > 1)$, so that no differential invariants exist.

9. In one of his papers Riquier has stated that a reduction of a system of partial differential equations to the first order would be advantageous. Mr. Robinson discusses what these advantages might be, making use of the principle of "correlation multiplicatoire." The reduction of a system to the orthonomic passive form, which has been left in an incomplete state by Riquier, is taken up in some detail.

10. The object of Professor Boutroux's paper is to distinguish a special family of rational differential equations of the first order and the first degree, and to show how the integrals of these equations can be investigated in respect to their general behavior (croissance), the number and the situation of their critical points, etc., and how the different branches of these integrals can be calculated by a process of successive approximations and represented by a single convergent series in the whole plane of the variable.

The simplest example of a differential equation of the family in question is provided by the equation

$$\frac{dy}{dx} = A_0 + A_1 y + A_2 y^2 + A_3 y^3,$$

which may be written

$$z\frac{dz}{dx} = A_3 + A_2 z + A_1 z^2 + A_0 z^3, \tag{1}$$

and which Professor Boutroux has already considered in several papers. Equation (1) belongs to our family when A_1 and A_0 are both equal to zero identically and the degrees m_3 and m_2, in x, of A_3 and A_2 satisfy the condition $m_3 > 2m_2 + 1$.

If we now consider the general rational equation of the first order and degree, it may be written

$$\frac{dz}{dx} = \frac{A_0 + A_1 z + \cdots + A_p z^p}{B_0 + B_1 z + \cdots + B_q z^q}, \tag{2}$$

where the A's and B's are polynomials in x, the respective degrees of which we shall call $m_0, \cdots, m_p$ and $n_0, \cdots, n_q$.

Equation (2) will belong to our family if the following conditions are satisfied:

$$p < q + 1; \quad m_0 - n_{q+1} > 0;$$

and

$$m_1 + \sigma < m_0, \cdots, m_p + p\sigma < m_0;$$

$$n_0 + \sigma - 1 < m_0, \cdots, n_{q-1} + q\sigma - 1 < m_0;$$

moreover, we suppose that there is no common root of the q polynomials $B_0, \cdots, B_q$.

11. Professor Fine's presidential address will appear in full in the April BULLETIN.

12. Dr. Hurwitz gives another proof of a theorem recently announced by Platrier (Paris thesis), which expresses minors of any order of the Fredholm determinant for the kernel of an integral equation in terms of the first minor and the determinant itself.

13. By means of a simple explicit reduction of the restricted problem of three bodies to the consideration of the transformation of a ring into itself, Professor Birkhoff establishes the existence of infinitely many symmetric orbits for certain ranges of values of the Jacobian constant, and orders these in four categories, each member of any category depending on two characteristic integers. A number of other results are also obtained.

A ring representation, but in Keplerian variables, was one of the great steps in advance accomplished by Poincaré (Les Méthodes nouvelles de la Mécanique céleste, volume 3, chapter 33). In a recent paper (*Rendiconti del Circolo Matematico di Palermo*, volume 33 (1912), pages 375–407) he indicated, on the basis of this representation, that if a certain geometric theorem (established by Professor Birkhoff, *Transactions*, volume 14 (1913), pages 14–22) were true, the existence of an infinite number of periodic solutions would follow for the above and similar problems; this fact had not been established by Poincaré in his earlier work, although it had been proved by him that for small enough values of one of the masses the number of periodic solutions was arbitrarily large.

Professor Birkhoff's present paper does not employ the geometric theorem of Poincaré but employs instead a certain

symmetry, in consequence of which the transformation of the ring is seen to be a product of two involutoric transformations.

14. In the contracted form of Horner's method, as described, for example, in Burnside and Panton's Theory of Equations, the contraction may be commenced at any time after the decimal portion of the root begins to appear, and the number of additional figures obtained by the contracted process will be one less than the number of figures in the trial divisor at the time the contraction commences. The question as to how many of the additional figures thus obtained will be reliable seems not to be discussed in any of the current text-books. Professor Huntington shows that the last n figures of the root as thus obtained may be erroneous, where n is the degree of the equation. As an example, he constructs an equation of the sixth degree, having the following coefficients: $a_0 = 500{,}000$, $a_1 = -3{,}295{,}000$, $a_2 = 9{,}047{,}550$, $a_3 = -13{,}249{,}720$, $a_4 = 10{,}914{,}563$, $a_5 = -4{,}794{,}993$, $a_6 = 877{,}562$. This equation has a root between 1 and 2. If we obtain the first two figures by the unabbreviated process and then as many more as possible by the contracted process, we shall find for the root the value 1.19 160 438. The last six figures in this result are wrong. The paper will be offered to the *American Mathematical Monthly*.

15. Professor Glenn considers invariants and covariants of binary forms whose coefficients are arbitrary variables, subject to transformations which are of the general binary linear type but whose coefficients are parameters representing integers reduced modulo p. These invariants were first defined by Hurwitz. The present paper deals with a method of finding systems of concomitants of this type. This is done by combining methods of Boole for systems now called Boolean, with results by Dickson in his memoir, "A fundamental system of invariants, etc.," *Transactions*, volume 12.

The paper contains tables of irreducible concomitants of the binary quadratic modulo 3, and of the binary cubic modulo 2. These concomitants are exhibited as transvectants taken with respect to the modulus. A section on general properties and the annihilators of this type of invariant is also included in the paper.

It is not yet known whether the formal modular concomitants of a form, as described above, constitute a closed system.

16. It is Professor Evans's object to express Green's function for the general linear partial differential expressions of the second order, in terms of Green's functions for the differential expressions consisting of the terms of the second order alone. These functions may be obtained by means of a system of linear integral equations, whose solutions may be written in closed form, according to a method first used by Hilbert. To carry out the analytical work effectively, however, it is desirable to consider these terms of the second order as a single differential operator, rather than as the sum of derivatives. This is, in fact, a point of view suitable to the applications in physics.

Such an interpretation is given by Professor Bôcher for the special case of Laplace's equation ("Harmonic functions in two dimensions," *Proceedings of the American Academy of Arts and Sciences*, volume 41). By means of this convention and a slight extension of the usual form of Green's theorem, the boundary value problems for the *non-homogeneous* equations may be solved without demanding so much as the finiteness and integrability of the non-homogeneous terms. The results for the non-homogeneous equations may then be applied to the solution of integro-differential equations of various types.

17. The differential equation considered by Professor Longley is $d^n y/dx^n = N/D$, where N and D are convergent integral power series in x and y, vanishing when $x = y = 0$. Similar singularities for a system of n equations of the first order have been treated by several writers* and from this general theory some results may be deduced concerning the equation above. In the present paper these results are proved directly and extension is made to some cases for which the general theory gives no information.

18. Dr. Graustein discusses the conditions under which two complex elements (points, planes, lines) of three-dimensional complex space may be real-congruent, i. e., equivalent under the group of real motions. The case of complex points

* An exposition of the results obtained, with some simplifications in method, is given by Dulac, *Bulletin de la Société mathématique de France*, vol. 40 (1912), pp. 324–383. The article contains references to several earlier papers.

is simple. Necessary though not in every case sufficient conditions for the real-congruence of two complex planes (or lines) result from the classification of complex planes (or lines) according to the number and situation of their real points and also according to whether or not they are minimal. To obtain necessary and sufficient conditions a complete system of absolute rational invariants of the complex plane (or line) in respect to the group of real motions is developed; the desired conditions may be stated in terms of these invariants. At the end of the paper the invariants of each complex element in respect to other real groups are discussed and from this the equivalence of two like named complex elements under these groups.

19. Dr. Reddick uses the power series method employed by Professor Kasner in dealing with conformal invariants of curvilinear angles (Proceedings of the Fifth International Congress of Mathematicians, volume 2, page 81). The transformations $Z = a_1 z + a_2 z^2 + \cdots$, where the coefficients are real, leave the X-axis invariant and form a subgroup of the entire conformal group. A classification of the invariants of an orthogonal net under this subgroup is given. Certain combinations of these invariants with those under the subgroup leaving the Y-axis invariant give invariants of the net under the entire conformal group.

20. Professor Osgood shows that Liouville's theorem, that a doubly periodic function of a single variable which is analytic for all finite values of the arguments is a constant, admits the following generalization:

Let $F(z_1 \cdots, z_n)$ be an integral function having in $(P_1^{(k)}, \cdots, P_n^{(k)})$, $k = 1, \cdots, r$, a primitive system of periods. Let the rank of the matrix

$$\left\| \begin{matrix} P_1' & \cdots & P_1^{(r)} \\ \cdot & \cdot & \cdot \\ P_n' & \cdots & P_n^{(r)} \end{matrix} \right\|$$

be denoted by ρ. Then $r \leqq \rho$.

A second theorem intimately related to the foregoing is as follows:

Let $(P_1^{(k)}, \cdots, P_n^{(k)})$, $k = 1, \cdots, r$, be r systems of complex quantities not satisfying a linear relation with real coefficients

$$c'P_l' + c''P_l'' + \cdots + c^{(r)}P_l^{(r)} = 0,$$

$$l = 1, \cdots, n, \text{ and } |c'| + \cdots + |c^{(r)}| > 0.$$

Let the rank of the matrix

$$\begin{Vmatrix} P_1' & \cdots & P_1^{(r)} \\ \cdot & \cdot & \cdot \\ P_n' & \cdots & P_n^{(r)} \end{Vmatrix}$$

be denoted by ρ. Let $F(z_1, \cdots, z_n)$ be an integral function admitting the r periods $(P_1^{(k)}, \cdots, P_n^{(k)})$. If $r > \rho$, then F is a constant.

21. There is given in n-space an arbitrary field of force in which the motion of a freely moving particle is defined by the equations

$$\frac{d^2x_i}{dt^2} = \psi_i(x_1, x_2, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

where the ψ_i are arbitrary functions of the n variables $x_1, \cdots, x_n$, possessing derivatives of the first and second orders in the region of space considered.

The object of Mr. Kells' paper is to find a set of purely geometric properties which completely characterize the system of curves belonging to the above defined field of force. The following four properties are found to be sufficient:

I. The osculating planes at a given point of all curves of the system through that point form a hyperpencil of planes in n-space, that is, all osculating planes at a point pass through a fixed line.

II. The centers of the third order osculating hyperspheres at a point of the ∞^1 curves passing through that point in a given direction lie in an $(n-2)$-flat.

III. The ∞^{n-1} $(n-2)$-flats which correspond according to property II to the ∞^{n-1} lineal elements at a point make up the secant $(n-2)$-flats of an nth order n-space curve which passes through the point in the direction of the axis of the hyperpencil of osculating planes at the point.

IV. Certain plane systems of curves associated with the $(2n-1)$-fold infinite system of curves in n-space are of the two-dimensional dynamical type.

(Professor Kasner has given the discussion for the cases $n = 2$ and $n = 3$. See *Transactions* for 1906 and for 1907.)

22. If one replace the E sets of the function $f(x)$, as defined by M. Lebesgue, by integrand sets similar to those used by W. A. Wilson, it is shown that these upper and lower functions must exist between every Baire class. In fact, their existence between every Baire class is a necessary and sufficient condition for the existence of functions of every Baire class. By the use of these intermediate functions it is possible to extend Baire's theorem on pointwise discontinuous functions to functions of every Baire class. Dr. Cramblet also proved that a function of class n defined over any set $\mathfrak{A}$ may be extended over a set $\mathfrak{B} > \mathfrak{A}$ without altering the class of the function. Finally, necessary and sufficient conditions that the limit of a sequence of functions of any class be a function of that class are proved, and are found to include Arzelà's convergence in segments as a special case.

23. In this note Dr. Lamond gives sufficient conditions for the existence and equality of the multiple and the iterated L-integrals of a function, over a field which may not be measurable.

24. The existence of certain relations between the first partial derivatives of a function of n independent variables is shown by Professor Howland to be a necessary and sufficient condition that it be a function of r combinations of these variables ($r < n$). The case where these combinations of the variables are linear is discussed at greater length. Two conditions for this case are derived, one involving the rank of a determinant, the other the rank of a matrix, and relations are shown to exist between the determinant and certain determinants of the matrix.

25. In Dr. Tracey's paper the plane rational curve of order n is given by the equation $(a\xi)(\alpha t)^n = 0$ and this is treated as a binary n-ic in the parameter t of the curve. Any covariant of this form involves the coordinates of the line ξ, and for a varying ξ becomes a class curve. If a covariant vanishes identically, a system of class curves is obtained all of which contain special line sections of the base curve, and in some cases the complete system of curves of a given order on these lines is at once determined.

A number of these covariant curves are discussed for the

rational quintic. The paper will be published in the *American Journal of Mathematics.*

26. The theory of restricted systems of equations has not been developed thus far in such a way as to permit of its application to many of the relatively simple problems of enumeration in geometry and algebra. The object of Professor Coble's paper is to remodel this theory and to increase thereby its utility.

27. A surface of Voss is characterized by the property that there exists upon it a one-parameter family of geodesics whose conjugate curves are geodesics also. This conjugate system has the same Gaussian representation as the asymptotic lines of a pseudospherical surface. Professor Eisenhart has established a transformation of surfaces of Voss which has the following geometrical properties: If S_1 is a surface resulting from S by such a transformation, the lines joining corresponding points of these surfaces form a congruence whose developables cut S and S_1 in the geodesic conjugate system. Moreover, the line of intersection of the tangent planes to S and S_1 at corresponding points generates a normal congruence, whose focal planes bisect the angles between these tangent planes; and the pseudospherical surfaces with the same spherical representation of their asymptotic lines as the conjugate system on S and S_1 can be brought into the relation of the transformation of Bäcklund. The analytical determination of the transformations requires the solution of a Riccati equation and quadratures. With each pseudospherical surface there are associated in the manner mentioned above a family of *special* surfaces of Voss, each of which admits a transform such that the lines joining corresponding points on the two surfaces are concurrent. These special surfaces play an important role in the general transformations of surfaces of Voss. It may be shown also that if S_1 and S_2 are two transforms of a surface of Voss S there exists a fourth surface S' which is a transform of both S_1 and S_2; moreover, S' can be found without quadratures.

28. In showing that the first derivatives of a certain nonlinear integral equation exist, Mr. Galajikian was led to the consideration of an equation of the type

$$u(x, x_0) = g\left\{x, \int_{x_0}^{x} f_1\left(x, t, \int_{0}^{t} u(t, \xi)d\xi\right) u(t, x_0)dt, \right.$$
$$\left. \int_{x_0}^{x} f_2\left(x, t, \int_{x_0}^{t} u(t, \xi)d\xi\right) u(t, x_0)dt \right\}.$$

In the present paper he proves, by a modification of Picard's approximation method, that a unique solution of this new equation exists in a sufficiently small interval.

29. In this paper, Dr. Gronwall considers systems of linear total differential equations in n independent variables such that their general solution depends linearly on a finite number of particular solutions. The coefficients of the system are assumed to be meromorphic. After a discussion of the decomposition of the singularities of the coefficients into irreducible manifolds, the main results of Fuchs's theory of linear differential equations are extended to the present case. Letting $\varphi(x_1, \cdots, x_n) = 0$, where φ is an entire function, be one of the irreducible singular manifolds, it is shown that each of a fundamental system of solutions may be expressed in the form: φ^r times a polynomial in $\log \varphi$, the coefficients of which are Laurent series of the kind considered in the following paper. The reduction of the system of equations to a canonical form is then taken up and applied to finding the necessary and sufficient conditions that the Laurent series mentioned shall contain only a finite number of negative powers of φ.

30. Dr. Gronwall obtains an extension of Laurent's theorem, which may be stated as follows, in the simplest case: Let $f(x_1, \cdots, x_n)$ be a uniform analytic function, holomorphic at all finite points except those satisfying the equation $\varphi(x_1, \cdots, x_n) = 0$, where φ is an entire function. Then

$$f(x_1, \cdots, x_n) = \sum_{\lambda=-\infty}^{\lambda=\infty} \frac{\psi_\lambda(x_1, \cdots, x_n)}{[\varphi(x_1, \cdots, x_n)]^n},$$

the ψ's being entire functions, and this series is uniformly convergent in any finite region, the points of which satisfy the inequality $|\varphi(x_1, \cdots, x_n)| > \epsilon > 0$. It is, however, essential to note that the above development is not unique, as in the case of one variable.

31. Dr. Jackson has shown (*Transactions*, October, 1912)

that, $f(x)$ being a function of period 2π and satisfying the Lipschitz condition

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$$

for all values of x_1 and x_2, then there exists, for every integer n, a trigonometric sum of order m $(2m - 2 \leqq n \leqq 2m)$,

$$T_n(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx$$

such that for all values of n

$$|f(x) - T_n(x)| \leqq \frac{J_m'}{J_m} \cdot \frac{\lambda}{n}.$$

Here J_m' and J_m are certain definite integrals, the quotient of which is bounded for all values of m, and by asymptotic considerations, Dr. Jackson finds an upper boundary for this quotient. In the present paper, Dr. Gronwall gives a direct proof that the quotient in question decreases as the integer m increases, and thus obtains a closer upper boundary.

32. The paper by Mr. Beetle establishes the following theorem:

In order that the osculating circles of a one-parameter family of curves on a surface form a cyclic system, it is necessary and sufficient that the curves be lines of curvature of constant geodesic curvature. If the geodesic curvature is zero, the curves are the plane lines of curvature of a surface of Monge. If the geodesic curvature is different from zero, the curves are the spherical lines of curvature on one of the surfaces S obtained from the surfaces of Monge by inversion, or on one of the surfaces obtained by subjecting the surfaces S to Bianchi's Combescure transformation* of surfaces with a system of spherical lines of curvature.

33. In the study of certain configurations by means of Professor Wilczynski's general method, the discussion is much simplified by throwing the general system of differential equations S into a canonical system S'. The reduction frequently requires the integration of partial differential equations, and so is of course (practically) impossible. Dr. Green

* Bianchi, Lezioni di Geometria differenziale, vol. 2, p. 306.

illustrates, by setting up a complete system of invariants and covariants for the general system of differential equations S of the theory of curved surfaces, a fact of which he has made use in previous papers (see abstracts in the January BULLETIN, pages 171, 172). The invariants and covariants calculated for the system S' may be expressed explicitly in terms of the coefficients of S, without actually requiring the integration necessary to throw S into the form S'. Not only is this procedure preferable on account of the simplifications which result from the use of the canonical form S', but it also gives an insight into the real nature of the invariants and covariants.

34. In a recent paper* Professor L. E. Dickson has discussed the algebra defined by the relations

$$xy = y\theta(x), \quad y^n = g,$$

where $\theta(x)$ is a rational polynomial in x in a given field F, and y^n is the first power of y which is commutative with x. In this note Professor Wedderburn gives a simple proof that, if g is properly chosen in F, this algebra is primitive, i. e., division by any number of the algebra except zero is always possible.

F. N. COLE,
Secretary.

WINTER MEETING OF THE SOCIETY AT CHICAGO.

THE thirty-second regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, December 26–27, 1913. In accordance with a recent action of the Council the meetings of the Chicago Section for the presentation of scientific papers are hereafter to be designated as meetings of the Society at Chicago, and this was the first occasion to be observed under the new arrangement. Seventy-three persons were in attendance, including the following fifty-six members of the Society:

Professor W. H. Bates, Professor G. A. Bliss, Dr. Henry Blumberg, Professor Daniel Buchanan, Professor W. H. Butts, Professor H. E. Cobb, Professor D. R. Curtiss, Professor S. C. Davisson, Professor L. E. Dickson, Professor Arnold Dresden,

* *Transactions*, Jan., 1914.

Professor Arnold Emch, Professor Peter Field, Professor W. B. Ford, Dr. Tomlinson Fort, Professor A. B. Frizell, Mr. M. G. Gaba, Professor A. G. Hall, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Professor T. F. Holgate, Mr. L. A. Hopkins, Professor L. C. Karpinski, Professor A. M. Kenyon, Professor C. E. Love, Professor A. C. Lunn, Professor W. D. MacMillan, Dr. H. F. MacNeish, Professor C. R. Mann, Professor J. L. Markley, Professor G. A. Miller, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor G. W. Myers, Mr. V. C. Poor, Professor H. L. Rietz, Dr. Mildred Sanderson, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Dr. H. M. Sheffer, Professor C. H. Sisam, Professor H. E. Slaught, Professor A. W. Smith, Mr. R. B. Stone, Professor A. L. Underhill, Professor J. N. Van der Vries, Professor E. B. Van Vleck, Miss Mary E. Wells, Professor Marion B. White, Professor E. J. Wilczynski, Professor F. B. Wiley, Dr. K. P. Williams, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

The reading of papers occupied four half-day sessions. Professor E. B. Van Vleck, President of the Society, occupied the chair on Saturday morning, and Professor Alexander Ziwet, past Vice-President, on Friday afternoon during the presentation of the retiring address by Professor D. R. Curtiss, chairman of the Chicago Section. Professor Curtiss presided during the rest of Friday afternoon and on Friday morning, and Professor E. J. Wilczynski, the newly elected chairman, on Saturday afternoon.

At the business meeting on Saturday morning the following officers of the Chicago Section were elected: Chairman, Professor E. J. Wilczynski; Secretary, Professor H. E. Slaught; and third member of the program committee, Professor A. M. Kenyon. The presentation of a formal paper by the retiring chairman has proved so satisfactory that the program committee was requested to arrange for more papers of this character; and since the chairmen now serve two years it was proposed to invite past chairmen to present such papers in the alternate years. Various suggestions were also referred to the committee for relieving the programs of their present congestion, particularly the plan of dividing the papers into groups and holding two simultaneous sessions during part of the meeting.

On Friday evening the University of Chicago tendered a complimentary dinner to the members of the Society and others attending the meeting. President Van Vleck presided on this occasion and Dean James R. Angell represented the University in the absence of President Judson from the city. There were informal addresses on the growth and influence of the American Mathematical Society from its inception in 1894, particularly with reference to the contribution of the West since the organization of the Chicago Section in 1896, by President Van Vleck, Professor Moore, first chairman of the Section, Professor Holgate, first secretary, and Professor Slaught, present secretary. These were followed by an address of welcome and congratulation combined in felicitous strain by Dean Angell. President Van Vleck expressed the gratitude of the Society for the cordial hospitality of the University of Chicago, and at the subsequent business meeting on Saturday morning he presented a formal resolution of thanks, which was adopted and transmitted to President Judson.

The following papers were presented at this meeting:

(1) Professor ARNOLD EMCH: "On the properties of certain systems of functions."

(2) Professor L. C. KARPINSKI: "The algebra of Rollandus."

(3) Professor W. D. MACMILLAN: "A reduction of certain analytic differential equations to differential equations of an algebraic type."

(4) Professor W. D. MACMILLAN: "On Foucault's pendulum."

(5) Miss OLIVE HAZLETT: "Invariants which characterize some linear associative algebras."

(6) Mr. A. R. SCHWEITZER: "Note on functional equations."

(7) Professor DANIEL BUCHANAN: "Periodic orbits on a smooth surface."

(8) Professor DANIEL BUCHANAN: "Periodic motion in a curve approximating a vertical parabola."

(9) Professor D. R. CURTISS: "Laguerre's extension of Descartes' rule of signs."

An interval was here devoted to informal questions and discussion.

(10) Mr. VINCENT C. POOR: "Some exact solutions of differential equations of motion of an incompressible viscous fluid."

(11) Professor G. A. MILLER: "A non-abelian group whose group of isomorphisms is abelian."

(12) Professor G. A. MILLER: "A group of order p^m whose group of isomorphisms is of order p^a."

(13) Professor G. A. MILLER: "A group admitting outer isomorphisms which do not permute any sets of conjugate operators."

(14) Professor F. R. MOULTON: "On the deviations of falling bodies."

(15) Professor E. J. WILCZYNSKI: "A new representation for a certain class of surfaces with indeterminate directrix curves."

(16) Professor C. E. LOVE: "On the irregular integrals of linear differential equations."

(17) Professor J. B. SHAW: "The two fundamental operations of general vector analysis."

(18) Professors G. A. BLISS and A. L. UNDERHILL: "The minimum of a definite integral for unilateral variations in space."

(19) Professor G. A. BLISS: "On the Weierstrass E-function."

(20) Dr. HENRY BLUMBERG: "A general theorem regarding sets of first and of second category."

(21) Dr. HENRY BLUMBERG: "The oscillation equation $\omega\omega f(x) = g(x)\omega f(x)$."

(22) Professor A. B. FRIZELL: "A non-enumerable well-ordered set."

(23) Professor E. J. MOULTON: "On figures of equilibrium of a rotating compressible fluid body."

(24) Professor L. E. DICKSON: "A linear associative division algebra in n^2 units."

(25) Professor L. E. DICKSON: "A form of Cayley's algebra from which its remarkable properties may be deduced at once."

(26) Mr. A. R. SCHWEITZER: "On quasi-transitive and symmetric functions."

(27) Mr. A. R. SCHWEITZER: "On the sensed and non-sensed point fields in the foundations of geometry."

(28) Professor PETER FIELD: "On constrained motion."

(29) Dr. K. P. WILLIAMS: "Concerning the linear difference equations of first order."

(30) Dr. C T. SULLIVAN: "Concerning a certain set of organic curves on a ruled surface."

(31) Dr. G. E. WAHLIN: "The binomial equation of prime degree in an arbitrary domain."

(32) Professor G. N. BAUER and Dr. H. L. SLOBIN: "Algebraic and transcendental numbers."

(33) Professor G. N. BAUER: "Conditionally convergent double series."

(34) Professor W. H. BATES: "The curvatures of a λ-space R_λ represented as differential parameters of an n-space containing R_λ."

Miss Hazlett was introduced by Professor Dickson. The papers of Mr. Schweitzer, Dr. Wahlin, Dr. Sullivan, Professor Bauer, and the second and third papers of Professor Miller were read by title.

Under the head of informal questions and discussions, the Secretary presented some data concerning the membership of the Society in which it appeared that of the approximately 600 colleges listed by the United States Commissioner of Education only about 150 are represented in the Society by members of their faculties; and the question was raised whether the Society's influence ought not to embrace at least all those institutions in which courses in collegiate mathematics, including the calculus, are taught.

Abstracts of the formal papers are given below, the numbers corresponding to those in the list of titles above.

1. In the theory of implicit real functions of several variables the cases considered are mostly those in which the corresponding Jacobians do not vanish simultaneously for a set of variables within the interval for which the functions are defined. On the other hand, when this happens, we may expect particular relations between the functions which deserve to be investigated. It is the purpose of Professor Emch's paper to show the importance of such cases by studying a certain system of functions and their Jacobians in connection with a geometrical problem. The paper will appear in the *Annals of Mathematics*.

2. In 1424 Rollandus, canon of St. Chapelle in Paris, wrote at the command of John, Duke of Lancaster, a summary of mathematics. This important work is preserved in manuscript, in a unique copy so far as we know, in the library of Mr. G. A. Plimpton. The treatise includes a discussion on

algebra which is found in the second section of the second part of the work and extends from folio 84′ to 110′. Professor Karpinski continues his series of studies on medieval algebra with an examination of this work. The study shows that Rollandus drew very extensively from the Quadripartitum Numerorum of Johannes de Muris whose work in algebra has been analyzed in an earlier paper of this series.* In many parts the work is largely a copy of the treatise which preceded it by a century. This mathematical treatise of Rollandus is of importance not only because it shows the influence of de Muris in France, but more particularly because it throws valuable light on the state of mathematical learning in France in the early fifteenth century.

3. Suppose that there is given a set of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

and that $X_i(0, \cdots, 0) = 0$; suppose further that the X_i functions are expansible in powers of the x_j, and that the x_j functions are normal variables, so that the equations can be written

$$\frac{dx_i}{dt} = \alpha_i x_i + \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \beta_{j_1}^{(i)} \cdots {}_{j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \quad \left\{ \begin{matrix} i = 1, \cdots, n \\ \Sigma_{k=1}^{k=n} j_k \geqq 2 \end{matrix} \right\}$$

then the question raised by Professor MacMillan is this: Can there be found a convergent substitution

$$x_i = y_i + \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \gamma_{j_1}^{(i)} \cdots {}_{j_n} y_1^{j_1} \cdots y_n^{j_n} \quad \left\{ \begin{matrix} i = 1, \cdots, n \\ \Sigma_{k=1}^{k=n} j_k \geqq 2 \end{matrix} \right\}$$

such that the resulting differential equations in the variables y_i

$$\frac{dy_i}{dt} = Y_i(y_1, \cdots, y_n) \qquad (i = 1, \cdots, n)$$

are algebraic, and in particular that the Y_i are polynomials in $y_1, \cdots, y_n$. In answer to the question it is shown that, if the constants α_i satisfy a very simple condition, such a substitution can be found for which the resulting differential equations are

* *Bibliotheca mathematica*, ser. 3, vol. 13, pp. 99–114 (May, 1913).

$$\frac{dy_i}{dt} = \alpha_i y_i \qquad (i = 1, \cdots, n).$$

In case the condition is not satisfied by all of the α_i but is satisfied by a group of them, say $\alpha_1, \cdots, \alpha_k$, then there exists a substitution

$$x_i = y_i + \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} \gamma_{j_1}^{(i)} \cdots {}_{j_k} y_1^{j_1} \cdots y_k^{j_k} \qquad (i = 1, \cdots, k),$$

$$x_i = 0 + \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} \gamma_{j_1}^{(i)} \cdots {}_{j_k} y_1^{j_1} \cdots y_k^{j_k} \qquad (i = k+1, \cdots, n),$$

for which the resulting differential equations are

$$\frac{dy_i}{dt} = \alpha_i y_i \qquad (i = 1, \cdots, k).$$

In the first case the transformed equations are equivalent to the original within the region of convergence of the substitution, for there is a unique correspondence between the values of the x_i and y_i in this region. In the second case the number of variables y_i is less than the number of the x_i so that such a complete correspondence is not possible. The cases where the above conditions are not satisfied lead to an extended field which has as yet been but slightly investigated by the writer.

4. No rigorous solution of the equations of motion of Foucault's pendulum has hitherto been given. In this paper by Professor MacMillan a rigorous solution is obtained by the methods of periodic orbits. For a given pendulum there are two arbitrary constants, namely, the amplitude of the oscillation and the azimuth of the initial plane of motion. In the present solution the amplitude of the oscillation is arbitrary, but the initial plane of motion is either the plane of the meridian or the prime vertical.

5. In this paper Miss Hazlett finds invariants which completely characterize linear associative algebras of two and three units with constants of multiplication and coordinates in the field C of all real and complex numbers. By invariant is here meant a single-valued function (in Dirichlet's sense of correspondence) of the constants of multiplication which takes the same value for two algebras of the same number of units and

having constants of multiplication and coordinates in any field F, if the two algebras be equivalent with respect to F. The invariants $I_1, \cdots, I_m$ are said to characterize completely the algebras of n units over a field F when each I_k has the same value for two algebras only when they are equivalent with respect to F.

For any field F the rational integral invariants and covariants of the characteristic equations and of the coefficients of the powers of ω therein (regarded as functions of the coordinates of the general number of the algebra) are invariants and covariants respectively of the linear algebra; and they are expressible as rational integral functions of the determinants and minors of the $2n^2$ linear equations which are necessary and sufficient conditions for the existence of a modulus.

In view of this, from Study's results in the *Göttinger Nachrichten* for 1889 it follows that, in the field C, the ternary linear associative algebras with a modulus are completely characterized by their rank and by the rank of the discriminant of the coefficient of ω in either characteristic equation. By a suitable reduction, it is clear that the binary linear associative algebras (with or without a modulus) are completely characterized by the identical vanishing or non-vanishing of the coefficient of ω and by the rank of the discriminant of the coefficient of ω^0 in either characteristic equation.

6. Mr. Schweitzer proves the following theorem: If

$$\phi\{x, \phi(y, z)\} = \phi\{z, \phi(y, x)\},$$
$$\phi\{f(x, y), y\} = \psi(x), \quad f\{\phi(x, y), x\} = \psi(y),$$

and

$$\phi\{\psi(x), y\} = \psi\phi(x, y),$$

then there exists a function $\chi(x)$ such that

$$\chi[\phi(x, y)] = \chi(x) + \chi(y), \quad \chi[\psi(x)] = \chi(x) + c$$

and

$$\chi[f(x, y)] = \chi(x) - \chi(y) + c,$$

where c is a constant.

7. In his first paper Professor Buchanan proves the existence, and gives a practical method for the construction, of

periodic orbits described by a particle which moves, subject to gravity, on a smooth surface. When the surface is a sphere the problem reduces to the well known problem of the spherical pendulum. Taking the general solutions of the spherical pendulum as derived in Chapter III of Professor Moulton's Periodic Orbits, which will appear shortly, Professor Buchanan determines periodic orbits described on the surface $S + \epsilon S' = 0$, where $S = 0$ is the equation of the sphere in the problem of the spherical pendulum, S' is an analytic function of the three rectangular coordinates, and ϵ is a variable parameter. The solutions are expansible as power series in ϵ in which the coefficients are trigonometric functions of the time; and the period is the same as that of the vertical motion of the spherical pendulum.

The equation of the surface employed in this paper is the same as that used by Poincaré in his memoir "Sur les lignes géodésiques des surfaces convexes."* Periodic orbits play the same rôle in this paper as closed geodesics in Poincaré's memoir.

8. Using the same method as in his first paper, Professor Buchanan determines a periodic solution for the motion of a particle which moves, subject to gravity, in the curve

$$x^2 - py + \epsilon f(x) = 0,$$

where ϵ is a parameter and $f(x)$ is analytic in x. For $\epsilon = 0$ the curve becomes a vertical parabola and a periodic solution can be derived by means of elliptic integrals. Then the analytic continuation of this solution is made with respect to ϵ as in the first paper.

9. The paper of Professor Curtiss is concerned with results which Laguerre deduced from his proof of the applicability of Descartes' rule of signs to infinite series. Recent completions of Laguerre's theorems are especially considered.

10. It appears that the most general solution of the differential equations of motion of an incompressible viscous fluid was obtained by Lamb for the case of steady motion when the inertia terms are neglected. A further solution for the case

* *Transactions*, vol. 6, No. 3 (July, 1905).

of small motion has also been deduced in a general form for spherical surfaces. The paper by Mr. Poor exhibits solutions for the cases of steady and non-steady motion without neglecting the inertia terms. These solutions are obtained by expanding the velocity and the average normal pressure in powers of an arbitrary parameter, thus reducing the problem to the solution of an infinite sequence of integral equations. The second approximation for the translation of a sphere through an infinite fluid at rest at infinity is deduced.

11. In his "Second report on recent progress in the theory of groups of finite order," published in this BULLETIN, volume 9 (1902), Professor Miller stated, on page 116, that "no one seems to have investigated the question whether a non-abelian group can have an abelian group of isomorphisms." In Hilton's Finite Groups, 1908, page 233, the question "Can a non-abelian group have an abelian group of isomorphisms?" is placed in the list of "a few interesting questions still awaiting solution."

In the present paper Professor Miller proves the existence of a non-abelian group of order 64 which actually has an abelian group of isomorphisms. This non-abelian group G can be generated by four operators s_1, s_2, s_3, s_4 which satisfy the following conditions:

$$s_1^8 = 1, \quad s_2^2 = s_3^2 = s_4^2 = 1, \quad s_2 s_1 s_2 = s_1^5, \quad s_3 s_1 s_3 = s_1,$$

$$s_4 s_1 s_4 = s_1, \quad s_3 s_2 s_3 = s_2, \quad s_4 s_2 s_4 = s_3 s_2, \quad s_3 s_4 s_3 = s_4.$$

The central of G is generated by s_1^4 and s_3, and the group of inner isomorphisms is of order 8 and of type (1, 1, 1). The three operators of order 2 in the central of G are invariant under its group of isomorphisms. The latter group is abelian, of order 128, and of type $(1, 1, 1, \cdots)$.

12. In the Appendix to Hilton's Introduction to the Theory of Groups of Finite Order, 1908, page 233, the following question is asked: "Can a group of order p^e have a group of automorphisms whose order is also a power of p?" Professor Miller proved the existence of a group of order p^9, p being any prime number whatever, whose group of isomorphisms has an order which is a power of p. This group is generated by three operators s_1, s_2, and s_3, of orders p^4, p^3, and p^2 respectively, which satisfy the following conditions:

$$s_1^{-1}s_3s_1 = s_1^{p^3}s_3, \quad s_1^{-1}s_2s_1 = s_2^{p^2+1}, \quad s_2^{-1}s_3s_2 = s_3^{p+1}.$$

It contains two characteristic subgroups of order p, generated by s_1 and s_3 respectively, and it involves at least one characteristic subgroup of every order which divides the order of the group. Its central is of order p^6 and the central quotient group is abelian and of type (1, 1, 1). The central involves the pth power of every operator of the group and it is generated by such operators.

13. In the second edition of Burnside's Theory of Groups of Finite Order, 1911, page 89, it is stated that "no case is known of an outer isomorphism which changes each conjugate set of operations into itself. On the other hand, it is still an open question whether or no such isomorphisms exist." In the present paper Professor Miller proves the existence of a group G of order 128 which admits such isomorphisms. This group may be generated by seven operators of order 2, which will be denoted by the symbols $t_1, t_2, t_3, s_1, s_2, s_3, s_4$.

The first three of these operators generate the central of G, which is of order 8. The properties of the other operators are defined by the following equations:

$$s_1s_2s_1 = t_1s_2, \quad s_1s_3s_1 = t_2s_3, \quad s_1s_4s_1 = t_3s_4,$$

$$s_2s_3s_2 = t_2t_3s_3, \quad s_2s_4s_2 = t_2s_4, \quad s_3s_4s_3 = s_4.$$

To prove that this group has an outer isomorphism which changes each set of conjugate operators into itself, it may be observed that the isomorphism which transforms s_4 into t_3s_4, and transforms each of the three operators s_1, s_2, s_3 into itself, is an outer isomorphism of G. On the other hand, it can be easily proved that this outer isomorphism transforms into itself each set of conjugate operators of G. See W. Burnside, *Proceedings of the London Mathematical Society*, volume 11 (1912), page 40.

14. The problem of the deviations of falling bodies was taken up by Professor F. R. Moulton because of the divergence of results which have been secured by different methods. A body dropped from a point P_0 near the surface of a rotating mass M, and moving subject only to the attraction of M, strikes its surface at P_1. A plumb-line suspended from P_0 touches the surface of M at P_2. The deviations of

the falling body are the distances of P_1 from P_2 expressed either in geocentric angular measure, or linear measure with the distances measured along the surface of the earth. It is shown how to locate P_1 and P_2 for any potential function. Application is made to the case where the potential function is expansible in zonal harmonics, and it is found that the meridional deviation is composed of two parts, one away from the equator and one toward the equator, of which the latter is numerically the more important.

15. At the summer meeting of the Society at Madison, Professor Wilczynski announced the result that the finite equations of surfaces whose directrix curves are indeterminate and whose asymptotic curves belong to linear complexes can be obtained by quadratures. He now finds that the equations of these surfaces, referred to their asymptotic lines, may be written in the still simpler form

$$(1)\qquad \begin{aligned} x &= -2(U' - V')(u - v) + 4(U + V), \\ y &= U' - V', \quad z = u + v, \end{aligned}$$

where U and V are arbitrary functions of the single variables u and v, and where U' and V' are the derivatives of these functions. To avoid the appearance of ruled surfaces, which are only in an improper sense solutions of the problem, the third derivatives of U and V must not vanish identically.

Equations (1) are related to the earlier equations, given at the summer meeting, in very much the same way as the formulae of Weierstrass for minimal surfaces are connected with those of Enneper.

As in that other theory, the surfaces (1) are algebraic if and only if the functions $U(u)$ and $V(v)$ are algebraic functions of their arguments. In the simplest case, when U and V are polynomials of the third degree, the resulting surface is a rational sextic with twisted cubic curves for asymptotic lines.

16. Professor Love's paper considers the existence and form of asymptotic integrals of the homogeneous linear differential equation of the nth order, valid for large positive values of the independent variable. The coefficients are assumed to be developable in asymptotic (or convergent) power series.

No restriction is laid upon the roots of the characteristic equation as regards their order of multiplicity, but a certain assumption of a different nature is made in order to reduce the algebraic difficulties of the analysis. This assumption is such that the familiar case of distinct roots is included as a special case of the problem treated. It is shown that if the characteristic equation has l distinct roots, the n fundamental solutions of the differential equation fall into l groups, in such a way that corresponding to a p-fold root there is a group of p fundamental solutions proceeding in powers of $(1/x)^{1/p}$.

17. Professor Shaw shows that the two fundamental operations in vector analysis of n dimensions lead to the progressive products of Grassmann and to products of reduced order, from which two kinds of products, the complete associative productive may be built up and defined. If one of the vectors is the symbolic Hamiltonian ∇, they lead to the extension of the divergence and curl.

18. For the problem of the calculus of variations with fixed end points, the curves which minimize an integral of the form

$$J = \int F(x, y, z, x', y', z')dS$$

have previously been studied, provided that the minimizing arc along which the integral J is taken is free to traverse any portion of space or is restricted to lie entirely within a given region R. It may happen, however, that while no extremal joining the two fixed end points exists entirely interior to a region R, yet there may be a minimizing curve consisting partly of arcs interior to the region and partly of arcs along its boundary. This gives rise to the problem of investigating the properties which characterize a minimizing curve lying partly on a given surface S, while the variations with which it is compared are restricted to lie on one side of the surface, a problem analogous to one in the plane which has been discussed by Professor Bliss. The paper of Professors Bliss and Underhill has to do with the problem of a minimum of the integral above with respect to unilateral variations in space, that is, variations which lie on one side of a given surface. The study of the problem is justified by the fact that the conditions found, especially those analogous to the

usual Jacobi condition, are not merely generalizations of the corresponding conditions in the plane, but present new characteristics of interest.

19. In a recent paper in the *Mathematische Annalen* (volume 73 (1913), page 596) Behaghel has deduced, for problems of the calculus of variations in space, an expression for the Weierstrass E-function which is analogous to a well-known formula for similar problems in the plane. The result which he obtains is correct, but in his proof he uses functions, in his notation A, B, C, which may become infinite even for very simple problems. For such cases his proof would not be valid. In the paper of Professor Bliss an expression of slightly different form is obtained for the E-function, which may be reduced to that of Behaghel by a simple transformation, and a discussion is given of some of the properties of the E-function resulting from the formula. The methods are applicable to space of any number of dimensions.

20. In this paper the fundamental domain is taken to be the linear continuum between $x = 0$ and $x = 1$. Every irrational point c in that domain may be represented as a continued fraction in the form

$$\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} \cdots + \frac{1}{c_n} + \cdots \tag{1}$$

where c_n $(n = 1, 2, \cdots, \infty)$ is a positive integer. The theorem proved by Dr. Blumberg is as follows:

Let (K) be any given condition by virtue of which the following functions are determined:

$$\begin{aligned} \bar{n} &= F(n, c_1, c_2, \cdots, c_n), \\ c_{n+1}' &= F_1(n, c_1, c_2, \cdots, c_n), \\ c_{n+2}' &= F_2(n, c_1, c_2, \cdots, c_n), \\ &\cdots\cdots\cdots \\ c_{n+\bar{n}}' &= F_{\bar{n}}(n, c_1, c_2, \cdots, c_n). \end{aligned} \tag{K}$$

$F, F_1, F_2, \cdots, F_{\bar{n}}$ are assumed to be single-valued and to take only positive integral values for the arguments in question. Otherwise these functions are arbitrary. Let S_1 denote the set of irrational points c for which in the representation (1) it ha ppens only for a finite number of values of n that

$$c_{n+1}' = c_{n+1},\ c_{n+2}' = c_{n+2},\ \cdots,\ c_{n+\bar{n}}' = c_{n+\bar{n}},$$

and let S_2 be the complementary set of irrational points; then S_1 is of first category and S_2 of second category.

A special case of the theorem is as follows: The set of points c for which

$$\frac{1}{c_1}+\frac{1}{c_2}+\frac{1}{c_3}+\cdots \tag{2}$$

is convergent is of first category; the set of points c for which the series (2) is divergent is of second category.

21. In a paper read at the November, 1913, meeting of the Southwestern Section of the Society, Dr. Blumberg gave certain properties of the oscillation function $\omega f(x)$. By the aid of these results, he is able to solve completely the oscillation equation

$$\omega\omega f(x) = g(x)\omega f(x), \tag{1}$$

where $g(x)$ is any given continuous function of x and $f(x)$ is to be determined. The equation

$$\omega f(x) = g(x)f(x)$$

may in a certain sense be reduced to an equation of the form (1).

22. The discriminating analysis by Brouwer in the November BULLETIN (pages 81–96) puts in evidence the rôle that awaits a non-enumerable well-ordered set obtained without the intervention of the continuum. In this paper Professor Frizell proves that Axioms T in his "Axioms of ordinal magnitudes," presented to the Cambridge Congress, yield a set that is not countable. The method of proof is based upon that used by Cantor in proving the continuum nicht-abzählbar. A corollary is that, unless the set in question possesses the potency of the continuum, we have here a non-enumerable well-ordered set of lower potency than the continuum.

23. In this paper Professor E. J. Moulton considers an application of integral equation theory to the theory of figures of equilibrium of a rotating compressible fluid mass. Certain negative theorems are first proved, which are closely

related to well-known theorems of Hamy, Volterra, and Vérronet. They allow generality in the surfaces in contrast with the generality of the density functions of the earlier theorems. Figures of equilibrium are next discussed as functions of a compressibility parameter. A new method is thus obtained for arriving at the Poincaré "forms of bifurcation." Finally the methods are applied to the determination of the theoretical shape of the earth. By comparing the observed with the theoretical figure it seems certain that the assumed Laplace pressure-density equation is not approximately satisfied in the earth's interior.

24. As a direct generalization of real quaternions, Professor Dickson considers the linear algebra with the n^2 units $i^s j^t (s, t = 0, 1, \cdots, n-1)$ and coordinates in a field F, where

$$\phi(i) = 0, \quad ji = \theta(i)j, \quad j^r = g,$$

g and the coefficients of the polynomials ϕ and θ being in F. Moreover, let $\phi(x) = 0$ be a uniserial abelian equation in F, i. e., an equation irreducible in F and having the roots

$$i, \ \theta(i), \ \theta^2(i) \equiv \theta[\theta(i)], \ \cdots, \ \theta^{r-1}(i) \qquad [\theta^r(i) \equiv i].$$

This associative algebra is obtained and discussed in the *Transactions* for January, 1914. For $n = 2$, we may take $\phi = x^2 - c$; then right hand and left hand division except by zero is always possible and unique (briefly, it is a division algebra) if and only if g is not the norm $e^2 - cf^2$ of any number $e + fi$ in the field $F(i)$.* The corresponding theorem for $n = 3$ is proved. Soon after the above algebra in n^2 units was communicated to Professor Wedderburn, he succeeded in proving that, for any value of n, it is a division algebra if and only if g is not the norm of any number in $F(i)$. (See abstract 34 on page 301).

25. Any number of Cayley's algebra† with the units $1, e_1, \cdots, e_7$ may be expressed in the form $q + Qe$, where q and Q (also r and R below) are real quaternions in the units $1, e_1, e_2, e_3$, while $e = e_4$. By use of the multiplication table

* *Transactions*, vol. 13 (1912), pp. 65, 66.
† *Coll. Math. Papers*, I, p. 127; XI, p. 368.

for Cayley's eight units, it can be verified with the exercise of patience that

$$(q + Qe)(r + Re) = qr - R'Q + (Rq + Qr')e,$$

where r', R' are the quaternions conjugate to r, R. Starting with this condensed formula as the complete multiplication table of the algebra, Professor Dickson finds at once that the norm of a product is the product of the norms of the factors ($qq' + QQ'$ being the norm of $q + Qe$). Also at once that right and left-hand division except by zero is always possible and unique. These proofs rest upon the simplest algebraic properties of quaternions. They are given in § 14 of Linear Algebras, in press for the Cambridge Tracts.

26. In his first paper Mr. Schweitzer proved the following theorem: If $f\{x, f(y, z)\} = f\{z, f(y, x)\}$ and $\phi\{f(x, y), y\} = \psi(x)$ and $f\{\phi(x, y), x\} = \psi(y)$ and $f\{\psi(x), \psi(y)\} = f(x, y)$, then there exists a function $\chi(x)$ such that $\chi[\psi(x)] = \chi(x) + c$, $\chi[f(x, y)] = \chi(x) - \chi(y)$ and $\chi[\phi(x, y)] = \chi(x) + \chi(y) + c$, where c is a constant. If in particular one assumes $\psi(x) = x$, that is, $c = 0$, then this theorem reduces to a theorem previously obtained by the author.

The preceding relations are satisfied by $\psi(x) = \sqrt{x^2 - 1}$, $f(x, y) = \sqrt{x^2 - y^2}$, $\phi(x, y) = \sqrt{x^2 + y^2 - 1}$ and accordingly the following problem is investigated: Given the quasi-transitive and symmetric functions $f(x, y)$ and $\phi(x, y)$, to determine the functions $\Omega_f(x)$, $\Omega_\phi(x)$, $\chi_f(x)$, $\chi_\phi(x)$ such that the relations

$$f(x, y) = \Omega_f\{\chi_f(x) + \chi_f(y)\} \cdot \Omega_f\{\chi_f(x) - \chi_f(y)\},$$

$$\phi(x, y) = \Omega_\phi\{\chi_\phi(x) + \chi_\phi(y)\} \cdot \Omega_\phi\{\chi_\phi(x) - \chi_\phi(y)\}$$

are satisfied. Reference is made to the article of Lelieuvre, *Bulletin des Sciences mathématiques*, series 2, volume 27, page 31.

27. In the *American Journal*, April, 1912, it is shown how to construct the systems ${}^2R_2^{(0)}$, ${}^3R_3^{(0)}$, ${}^4R_4^{(0)}$, etc., as projective analogues of the systems ${}^1R_1$, ${}^2R_2$, ${}^3R_3$, etc. It is also shown, l. c., that the systems ${}^2R_2$ and ${}^2R_2^{(0)}$ may be generalized into a single system, say, ${}^2R_2^{(1)}$. Analogously the systems ${}^3R_3$ and ${}^3R_3^{(0)}$, ${}^4R_4$ and ${}^4R_4^{(0)}$, etc., may be generalized into systems

${}^3R_3^{(1)}$, ${}^4R_4^{(1)}$, etc., respectively. An n-dimensional point field Mr. Schweitzer defines to be the set of points satisfying (at least) the system ${}^nR_n^{(1)}$ $(n = 2, 3, \cdots)$. For example, a two-dimensional point field is given by the set of points in the interior and on the boundary of a circle of finite radius $\neq 0$ in the euclidean plane, since this set satisfies (*American Journal*, l. c.) the system ${}^2R_2^{(1)}$. Similarly, the set of points in the interior and on the boundary of a hypersphere of finite radius $\neq 0$ in euclidean n-dimensional space $(n = 2, 3, \cdots)$ satisfies the system ${}^nR_n^{(1)}$. Essentially two categories of point fields are to be distinguished, the *sensed* and the *non-sensed*. The latter are easily defined in terms of the former.

Applying the preceding conceptions to certain investigations* of E. Müller, the author constructs an $(n + 1)$-dimensional geometry $(n = 2, 3, \cdots)$ consisting of (1) the set of all circles (boundary and interior) of finite radius $\neq 0$ in the euclidean plane; (2) the set of all spheres (boundary and interior) of finite radius $\neq 0$ in the euclidean three-space, and so on. In particular, the sets of circles, spheres, etc., of the same† finite radius $\neq 0$ are investigated in each geometry. Two cases are to be distinguished according as a definite sense is or is not attached to the circular, spherical, etc., fields. In the non-sensed case, the set of all circular fields in the euclidean plane, gives rise to a three-dimensional hyperbolic geometry; similarly, the set of all non-sensed spherical fields in euclidean three-space gives rise to a four-dimensional hyperbolic geometry. When circular (spherical) fields are taken of the same radius there arises a limit surface abstractly identical with a euclidean plane (three-space).

Since in the preceding $(n + 1)$-dimensional geometry the elements are n-dimensional point fields, the geometry may be regarded as a theory on a class of relations $\{{}^nR_n^{(1)}\}$. This is compared with r-dimensional geometries constructed on the basis of a class of relations $\{{}^rR_r\}$, where $r < n$.

28. Recent discussion regarding the laws of sliding friction makes problems in constrained motion of special interest. Professor Field considers the motion of two particles which are connected by a rigid weightless rod, one or both of the

* E. Müller, Dissertation (Königsberg), where further references will be found.

† Cf. Müller, l. c., p. 279, (*c*).

particles being constrained to move on a rough curve or surface. The paper is devoted to a study of what the initial conditions must be in order that the equations of motion may give rise to (*a*) conditions which are compatible, (*b*) unique values for the accelerations involved.

29. The purpose of Dr. Williams's paper is to obtain a simple solution of the equation

$$f(x+1) = E(x)f(x),$$

where $E(x)$ is an entire or a meromorphic function. The method employed makes use of the Weierstrass factor form of $E(x)$. Barnes has also given a treatment of the equation by making use of the factor form of $E(x)$. His solution, however, contains certain complicated functions of the zeros and poles of $E(x)$, for which he was able to obtain the explicit form only in special cases. Dr. Williams proceeds in a way different from that used by Barnes, and obtains a solution which is completely determined. The method is also simpler, and exhibits in a clear way why certain convergence factors can be easily obtained.

30. In this paper Dr. Sullivan discusses a net of directrix curves which are related to a ruled surface in much the same way that the directrix curves introduced by Professor Wilczynski* are related to a curved surface. These curves and their associated linear congruences are completely determined by certain fundamental invariants h, θ', C'. When the twisted asymptotic curves belong to linear complexes the directrix curves form a conjugate system, and they can be found by quadratures. Theorems analogous to those on pages 98, 118, 119, 120 of the second memoir have been found for ruled surfaces. The directrix curves are indeterminate and the corresponding congruences degenerate, if and only if the invariants h, θ', C' vanish, that is, if and only if the surface is a Cayley cubic scroll. In this case the directrix congruences degenerate into the net of lines in the uniplane, and the sheaf of lines through the unode of the surface.

31. The first part of the paper by Dr. Wahlin is a development of some general properties of relative p-adic domains,

* Second memoir, *Transactions*, vol. 9 (1908).

and equations in an arbitrary p-adic domain. The main theme of the paper is, however, the determination of the nature of the p-adic domains in which the binomial equation of degree l, l being a prime, has l solutions. The equation is considered for the domains of the various prime divisors of an arbitrary algebraic domain, and in each case it is determined what the simplest irrationality is which must be adjoined to the given domain, in order that the equation in question may have l solutions.

32. In this paper Professor Bauer and Dr. Slobin investigate the relation of the argument of a trigonometric function to the function from the standpoint of the nature of the number represented. They set up various forms of the argument for which the function always represents algebraic numbers, and other forms for which the function represents transcendental numbers. It is shown that the trigonometric functions represent algebraic numbers if the argument is a rational multiple of π. A symbolic method is indicated whereby the algebraic equation may be set up for roots of the indicated type. The following is representative of the theorems developed: If a trigonometric function represents a number of the form $a\pi$, where a is an algebraic number other than zero, the argument is of the form $\log t\pi$, where t is a transcendental number.

33. The definition of the sum of a double series given by Pringsheim admits of conditionally convergent double series, while the definition of convergence given by Jordan does not. Adopting Pringsheim's definition, Professor Bauer shows that in a conditionally convergent double series

$$\lim_{p=\infty,\, q=\infty} S_{pq} = \lim_{m=\infty} S_{mm}.$$

This leads to a definition of multiplication for single series whereby two conditionally convergent series may be multiplied together with the assurance that the limiting sum of the resulting series will be equal to the product of the limiting sums of the given series. It is also shown that a conditionally convergent double series whose terms tend to zero may be rearranged so that the limiting sum of the rearranged double series is equal to any preassigned quantity.

34. This paper is a more general treatment of the subject presented to the Society by Professor Bates in April, 1910. In the former paper, only one value of λ (namely, $n-1$) was considered.

H. E. SLAUGHT,
Secretary of the Chicago Section.

SHORTER NOTICES.

Leçons sur le Prolongement analytique. By LUDOVIC ZORETTI. Paris, Gauthier-Villars, 1911. vi+115 pp.

IT is the plan of this little volume, like that of its fellows of Borel's excellent collection of monographs on the theory of functions, to conduct a student from a presupposed minimum of knowledge—but hardly of capacity—to the actual present frontier of science. This is evidenced by the many questions raised, but not to-day answerable, to be found scattered through its pages. It consists, in substance, of lectures delivered by its author at the Collège de France during the year 1908–1909; while it may be said to have for its subject such questions of the theory of functions as can be best treated, not by the methods of Cauchy and Riemann, but by going back to Weierstrass's fundamental definition of an analytic function. It is in this sense that the title is to be understood. Weierstrass's definition gives rise, in the general case, to a function with an infinite number of branches. The small advance that has been made in the study of such functions, the importance of which is incontestable (in the analytic theory of differential equations, for instance), is due, in part, to its difficulty, in part to the lack of appropriate tools. In particular—it is our author's idea—progress in the theory of these functions depends on progress in the theory of the Riemann's surface of an infinite number of sheets (as this latter is conceived of by Poincaré in his paper on the uniformization of functions, *Acta Mathematica*, volume 31, 1907). "The object of this book will have been accomplished if I succeed in persuading the reader of the great interest attaching to investigations of these surfaces" (page 43).

The work begins, after an interesting and suggestive introduction of six pages, with a chapter of 22 pages on (closed)

sets of points in one and two dimensions. The object here was to develop for use later certain properties which could not be considered to form part of the "classic" theory of the subject.

Chapter II, on the general idea of an analytic function, occupies more than a third of the book. A start being made from Weierstrass's definitions of analytic continuation and analytic function, some of the complexity of these notions is brought out by examples; as, for instance, that a point may be a regular point for each branch of a function, yet itself be a point of condensation of singular points.—What kind of set do the singular points of a function constitute? The answer to this question is, in the case of a one-valued function, that it may be any given (closed) set. Similarly a many-valued function exists having for its singular points any given enumerable infinity of closed sets; but whether this is the most general case is not known (see, however, the note on page 51). In attempting to extend to many-valued functions the idea of a natural boundary, we are met by the fact that a function may have a singular line which is not a natural boundary for any of its branches: this is exhibited by an example in the note at the end of the volume. A function may even have a limited domain of existence although no one of its branches has a natural boundary; this phenomenon our author expresses himself as "forced to believe in, but unable to explain." A distinction has then to be made between a natural boundary for a branch of a function and one for the function itself.

Singular points may be classified, not only by the kind of set they belong to, but secondly by the behavior of the function there. This introduces us to Painlevé's notion of the domain of indetermination of such a point.* The domain of indetermination of a point viewed as a singular point of a given branch of a function may be roughly described as the set of values of the function that may be reached by analytic continuation from the given branch without leaving an infinitesimal neighborhood of the point. Then transcendental singularities may be classified as follows: *ordinary* if the domain of indetermination be a point ($z = 0$ for $\log z$), otherwise *essential*; the domain of indetermination in this latter case being either *complete*, where it comprises the whole plane (the isolated essentially singular point of a one-valued function), or else *incomplete*.

* *Comptes rendus*, vol. 131 (1900).

Chapter III takes up Weierstrass's theorem as to the behavior of a one-valued function in the neighborhood of an (isolated) essential singularity—with a proof by the methods, based on analytic continuation, used in this book—and Picard's theorem.

Can Weierstrass's theorem be extended to other kinds of singular points (of a one-valued function)? This question is treated in Chapter IV. Intermediate between the isolated singular point and the singular line comes a point "belonging to a discontinuous set" (phrase exactly defined pages 54, 16). For certain of these cases, too, Weierstrass's theorem holds. But the attempt at a proof in the general case (sketched by Painlevé in his Stockholm lectures) failed; while the proof given by our author in his thesis that such a point was, at any rate, an essential transcendent singularity has turned out to be unsound. It now appears that, in point of fact, not even this narrower proposition is true; for Denjoy has produced an example of a function whose singular points are all of the kind in question, that is, each belongs to a discontinuous set, while the function, nevertheless, remains continuous throughout the plane; so that these transcendental singularities are all merely ordinary ones.

For one-valued functions with singular lines general results are almost non-existent. Chapter V deals with such functions and raises questions like the following. It is known that a function continuous in a region cannot have an *isolated* singular line there: can we say more, that a function with nothing beyond singularities where it remains continuous (ordinary transcendent singularities) is necessarily holomorphic?

Chapter VI concerns itself with an important class of many-valued functions, namely, the inverses of one-valued, and, in particular, of integral functions. The singular points of such a function can be at most of the ordinary transcendent variety. A property pretty much characteristic of them is that at least one of the areas in the plane of the function corresponding to a small circle, in the plane of the independent variable, about the singular point includes the point at infinity.

The reviewer might mention a couple of unimportant slips that have come under his notice: page 1, note 2, the reference should be to the *Monatsberichte;* page 43, note 2, the title of Boutroux's book is incorrectly given.

FRANK IRWIN.

Serret's Lehrbuch der Differential- und Integralrechnung. Vierte und fünfte Auflage, zweiter Band, bearbeitet von GEORG SCHEFFERS. Leipzig, Teubner, 1911. xiv+639 pp.

THE present edition (or let us say issues, since it seems to be dual!) is a reprint, with slight alterations, of the third. Besides the half dozen paragraphs listed in the preface there are three or four more that have undergone some considerable degree of remodelling. This adds nine pages to the bulk of the book. (The convenience of the student has been consulted, in making these alterations, by retaining, with few exceptions, the old paragraph numbering.) Beyond this the changes are those of phraseology.

The novelty of the present edition is an appendix of 46 pages of historical notes—to both the first and second volumes. It is purposed to supply the third volume also, whenever a new edition becomes necessary, with a like apparatus. These notes are in response to wishes often expressed and are welcome. They seem particularly full on matters of mathematical nomenclature, notation, symbols. The desirable custom has been followed of adding to the name of each (no longer living) author the years of his birth and death.

The reference on page 447 to *page* 405 should be to the paragraph of that number (*No.* 405).

FRANK IRWIN.

Ueber die Theorie benachbarter Geraden und einen verallgemeinerten Krümmungsbegriff. Von W. FRANZ MEYER. Leipzig und Berlin, Teubner, 1911. xi+152 pp.

THIS supplement to the ordinary text-book on differential geometry has, as its fundamental idea, a generalization of the notion of curvature of a space curve obtained by replacing the tangent at a point of the curve by an arbitrary line through that point. The idea is not new. Articles dealing with certain phases of the subject had appeared five years before the publication of this book. From time to time other papers were published by the author and his students and various other writers. Here these results are exhibited in their relations to one another and are supplemented by new material.

The book opens with a concise treatment of the moving trihedral, curvature, torsion, the Frenet formulas, etc., with reference to an arbitrary line g through a point of the space

curve. Then follows a systematic study of the shortest distances associated with two neighboring trihedrals. From convenient forms for the two principal formulas for the shortest distance between two skew lines are deduced the shortest distances (1) between two neighboring positions of g, (2) between a point P of the space curve and the line of striction of the ruled surfaces formed by the lines g, and (3) between P and the "normal plane" of a neighboring point on the curve. Six other formulas are obtained by substituting for g the binomial and the principal normal. The relations among these nine distances become noteworthy when the general trihedral is replaced by the canonical. Another special case is that in which g is required to be perpendicular to the tangent to the curve. This case becomes more important when the further restrictions are imposed that the curve shall lie on a certain surface and that g shall fall along the positive direction of the normal to the surface at the point. An appendix, which occupies one-third of the book, deals with generalized curvature in R_n and with the invariant representation in R_3.

E. B. COWLEY.

Grundlagen der Geometrie. Von DAVID HILBERT. Fourth edition. Leipzig, Teubner, 1913. vi+258 pp.

THE system of axioms of geometry presented in this new edition of Hilbert's classical work differs from that given in the third edition (1909) in regard to the third group of axioms, those of congruence, the number of which is now reduced from six to five. In the third edition, axiom III: 5 reads: When an angle (h, k) is congruent to both the angle (h', k') and the angle (h'', k''), then the angle (h', k') is congruent to the angle (h'', k''). In the present edition, this is now stated as theorem 10, and for a proof based upon the axioms in groups I and II and the remaining five axioms in group III, the reader is referred to a paper by A. Rosenthal: "Vereinfachungen des Hilbertschen Systems der Kongruenzaxiome," *Mathematische Annalen*, volume 71 (1912), pages 257–274.

A few other, but relatively unimportant, changes have been made in the text, and references to recent literature have been added.

T. H. GRONWALL.

Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. Von DAVID HILBERT. Leipzig und Berlin, Teubner, 1912. xxiv + 282 pp.

THIS book appears as volume 3 of a new Teubner collection entitled Fortschritte der mathematischen Wissenschaften in Monographien, under the general editorship of Professor Otto Blumenthal, and consists of a reimpression, with some insignificant changes, of the six papers on integral equations published by Hilbert in the *Göttinger Nachrichten* from 1904 to 1910, with an additional chapter on the foundations of the kinetic theory of gases (also published in *Mathematische Annalen*, volume 72). These various papers not having been remoulded into an organic whole, the presentation of the theory is not a systematic one, but this defect is largely offset by an elaborate table of contents arranged by subjects and stating the main theorems in extenso. Since Hilbert's papers have already become classical and are familiar to all students of the subject, an enumeration of the section headings will suffice: I. General theory of linear integral equations; II. Application of the theory to linear differential equations; III. Application of the theory to problems in the theory of functions; IV. Theory of functions of an infinite number of variables; V. A new exposition and extension of the theory of integral equations, VI. Application of the theory to various problems in analysis, geometry, and theory of gases.

T. H. GRONWALL.

Gabriel Lippmann. Par E. LEBON. Paris, Gauthier-Villars, 1911. viii+70 pp.

THE series Savants du Jour by Lebon has made known to mathematicians the world over, and let us hope to a general public in France, the portrait and the details of the life and work and writings of Poincaré, Darboux, Picard, and Appell. The attention is now turned to the physicist Lippmann. To one at all familiar with the rigorous program of studies and examinations through which the young French savant passes it is particularly interesting to read how Lippmann by too great an enthusiasm for extra-curriculum studies failed to pass the aggregation, took his doctorate in Germany, returned for his doctorate at the Sorbonne, and then marched steadily on to the leading position in French physics. In regard to the

biography as a whole, we may quote from the remarks of Darboux when presenting the work to the Académie des Sciences: " Cette Notice nouvelle est composée avec le même soin, avec le même souci de l'exactitude et selon la même méthode que les Notices précédemment parues. . . . M. Ernest Lebon ne néglige pas de nous faire connaître la genèse des plus belles découvertes de Gabriel Lippmann. . . . Nous n'hésitons pas à prédire à cette nouvelle Notice le succès et la faveur qui ont accueilli les précédentes."

E. B. WILSON.

College Mathematics Notebook. By R. E. MORITZ. Boston, Ginn and Company.

College Engineering Notebook. By R. E. MORITZ. Boston, Ginn and Company.

TWO notebooks, each $8\frac{1}{2} \times 10\frac{3}{4}$ inches, each of weight $1\frac{3}{4}$ pounds, in biflex binder with removable sheets, have been designed by R. E. Moritz and published by Ginn and Company. The first, for classes in advanced algebra, trigonometry, analytical geometry, and calculus, has 3 pages of simple formulas, 95 pages of rectangular coordinate paper ruled about 25 to the inch, 5 pages of polar coordinate paper ruled about $7\frac{1}{2}$ to the inch (radially), 2 pages containing seven 2-place tables, and one page of common graphs. The second, for classes in technical schools and colleges, contains 11 pages of simple mathematical and engineering formulas, the same outfit (apparently) of coordinate paper plus 5 pages of logarithmic paper, the same graphs, the same tables plus 4-place logarithms, 3-place natural functions, and a conversion table for circular measure.

For teachers who like to use notebooks of this type, these seem as elaborate as any. In so far as a notebook induces the student to greater system and neatness it is desirable. There are, however, undesirable features in a system of instruction which uses notebooks. For instance, in most work in mathematics a rough sketch of a curve is as good as a careful graph; using finely ruled paper generally leads the student to take the large amount of time necessary to construct a good graph or gives him too high an opinion of the accuracy of a hasty sketch; there are difficulties of the sort experienced in using 7-place tables on 3-place data; a coarser notebook might

be preferable. In so far as a reference table for formulas expedites a student's work, it is good; but if it leads him away from using the small amount of memory and common sense requisite for remembering most of the very simple and fundamental formulas here given, it is not good.

E. B. WILSON.

Gravitation. By FRANK HARRIS. London, Longmans, Green and Company, 1912. xi + 107 pp.

WHEN sending for this book, we hoped to find in it an account of past and present speculation concerning gravitation—especially in view of the recent active discussion under the lead of Einstein; we find, however, an independent theory of the hydrodynamic type with apparently no reference to the numerous prior authors who have tried to give a similar explanation of gravitational or electric phenomena. With picturesque language, guided by an entertaining philosophy, and frequently plodding through tedious approximations (the details of which we have not checked), the author discusses the mutual forces developed by two equal solid spheres moving in a perfect fluid with uniform velocity on the line joining their centers or in the line perpendicular thereto. A more general problem than this is far better treated by Lamb in his Hydrodynamics (arts. 97, 98, 136 of the second edition) with references to a fuller treatment.

Harris also treats sources and sinks and undertakes to make clear what is meant by the "cruelly ill-used" expression potential energy. In the last chapter of the work he argues for a higher dimensionality for space, accompanied by an extreme and progressive anisotropy in so far as the distribution of matter is concerned. Indeed our material universe now lies chiefly in the galactic plane, and is supposed to have become so completely three-dimensional that our sense of higher dimensions is atrophied (if it ever existed). These speculations may be interesting; but, unless supplied with more through analysis, they must be questioned.

E. B. WILSON.

NOTES.

THE opening (January) number of volume 36 of the *American Journal of Mathematics* contains the following papers: "Determination of all primitive collineation groups in more than four variables which contain homologies," by H. H. MITCHELL; "On non-homogeneous equations with an infinite number of variables," by R. D. CARMICHAEL; "On constrained motion," by P. FIELD; "Covariant curves of the plane rational quintic," by J. I. TRACEY; "The group of isomorphisms of an abelian group and some of its abelian subgroups," by G. A. MILLER; "Self-projective rational curves of the fourth and fifth orders," by R. M. WINGER; "Iterated limits in general analysis," by R. E. ROOT.

The frontispiece of the volume is a portrait of W. STAHL.

AT the meeting of the London mathematical society held on January 15 the following papers were read: By S. T. SHOVELTON, "A generalization of the Euler-Maclaurin summation formula," and "Deduction of formulas of mechanical quadrature from the Euler-Maclaurin summation formula"; by A. YOUNG, "Binary forms."

AT the Atlanta meeting of the American association for the advancement of science Professor E. B. VAN VLECK delivered a vice-presidential address before Section A on "The influence of Fourier's series upon the development of mathematics." The address has been published in *Science*, January 23. Professor H. S. WHITE was elected vice-president of Section A. The next meeting of the Association will be held at Philadelphia from December 28, 1914, to January 2, 1915, under the presidency of Dr. C. W. ELIOT. Professor F. R. MOULTON is secretary of Section A.

THE editors of volume 7 of the Encyclopedia of mathematical sciences will hold a joint session with the French philosophical society April 6–8, for the purposes of formulating the general plan of the work. Addresses or reports have already been arranged for, by the following speakers: Professors Enriques, Whitehead, Hadamard, Langevin, Couturat, Brunschwieg, Dufumier, Winter, and Reymond. The sessions will be held at the Sorbonne in Paris, under the presidency of Professor E. Boutroux.

THE tercentenary of the publication of JOHN NAPIER'S Logarithmorum Canonis mirifici Descriptio will be celebrated by a congress to be held at Edinburgh, beginning July 24, under the auspices of the Royal society of Edinburgh and a committee representing the Royal society of London and other scientific bodies. The proceedings of the congress will be published in a memorial volume. A general invitation is extended to all mathematicians and others interested to attend the celebration. Further information may be obtained from the general secretary of the Royal society of Edinburgh, 22 George Street, Edinburgh.

THE academy of sciences of Paris announces the following subjects for the next awards of the mathematical prizes: Bordin prize (3000 francs): for 1915, "A notable contribution in the investigation of curves of constant torsion; determine, if possible, such curves that are algebraic or at least those that are rational." As the prize was not awarded in 1913, the subject announced for that year will be repeated for the award of 1917: "Perfect in some important point the arithmetic theory of non-quadratic forms." Grand prize (3000 francs): "Apply the methods of Henri Poincaré to the integration of some simple linear algebraic differential equations." In addition to manuscripts submitted in competition the academy reserves the right to consider memoirs already published. The award will be made in 1916. The Poncelet prize (2000 francs) for a meritorious work in applied mathematics in 1915 and in pure mathematics in 1916. Vaillant prize (4000 francs) for 1917: "Determine and study all the surfaces which can be generated in two different ways by the motion of a rigid curve."

PROFESSOR O. PERRON, of the University of Tübingen, has accepted the professorship of mathematics at the University of Heidelberg, as successor to Professor L. KOENIGSBERGER.

PROFESSOR P. STROOBANT, of the University of Brussells, and Professor M. STUYVAERT, of the University of Ghent, have been elected members of the royal Belgian academy of sciences.

PROFESSOR A. KRAZER, of the technical school at Carlsruhe, has been elected an honorary member of the Swiss society of naturalists.

PROFESSORS A. GUTZMER, of the University of Halle, L. KOENIGSBERGER, of the University of Heidelberg, and R. VON LILIENTHAL, of the University of Münster, have received the title of Geheimer Regierungsrat.

DR. O. TOEPLITZ, of the University of Göttingen, has been appointed associate professor of mathematics in the University of Kiel.

DR. H. F. BAKER, Cayley lecturer in mathematics at Cambridge University, has been appointed Lowndean professor of astronomy and geometry, as successor to the late Sir ROBERT BALL.

THE Sylvester medal of the Royal society has been awarded to Dr. J. W. L. GLAISHER.

DR. C. M. WOODWARD, emeritus professor of mathematics and applied mechanics in Washington University, died on January 12, at the age of 77 years.

MR. D. L. PETTEGREW, of the Massachusetts mutual life insurance company, died on January 19, at the age of 86 years. Mr. Pettegrew had been a member of the American Mathematical Society since 1893.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BENDT (F.). Grundzüge der Differential- und Integralrechnung. 5te Auflage, durchgesehen und verbessert von G. Ehrig. Leipzig, Weber, 1914. 8vo. 16+268 pp. Cloth. M. 3.00

BERGMANN (H.). Das Unendliche und die Zahl. Halle, Niemeyer, 1913. 8vo. 7+88 pp. M. 2.50

BONNAR (W.). The mathematical laws of psychic phenomena; discovered, formulated and elucidated with practical diagrams. Buffalo, N. Y., Bonnar, 1913. 4to. 20+193 pp. $5.00

BOSMANS (H.). Les démonstrations par l'analyse infinitésimale chez Luc Valerio. (*Annales de la Société scientifique de Bruxelles, tome 37.*) Louvain, Ceuterick, 1913. 22 pp. Fr. 2.00

——. Sur quelques exemples de la méthode des limites chez Simon Stevin. (*Annales de la Société scientifique de Bruxelles, tome 37.*) Louvain, Ceuterick, 1913. 33 pp. Fr. 2.00

BÜCHER, Neue, über Naturwissenschaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels, nach Wissenschaften geordnet.) Mitgeteilt Herbst 1913. Leipzig, Hinrichs, 1913. 8vo. Pp. 51–74. M. 0.30

EHRIG (G.). See BENDT (F.).

FISCHER (A.). Ueber eine zyklische Fläche vierter Ordnung. (Diss.) Bern, 1913. 8vo. 52 pp.

FORT (O.) und SCHLÖMILCH (O.). Lehrbuch der analytischen Geometrie. 2ter Teil: Analytische Geometrie des Raumes. 7te Auflage, bearbeitet von R. Heger. Leipzig, Teubner, 1913. 8vo. 8+326 pp. Cloth. M. 6.80

GÖHNER (O.). Ueber Systeme algebraischer Korrespondenzen. (Diss.) Tübingen, 1913. 8vo. 53 pp.

GULDBERG (A.). See VERZEICHNIS.

HARTNUSS (O.). Anwendung der Pfaffschen Aggregate auf die orthogonale Determinante B. (Diss.) Giessen, 1912. 8vo. 53 pp.

HARZER (W.). Bestimmung der Anzahl der dreiblättrigen Riemannschen Flächen mit beliebig gegebenen Windungspunkten und der vierblättrigen mit Windungspunkten gleicher Ordnung. (Diss.) Leipzig, 1913. 8vo. 43 pp.

HEGER (R.). See FORT (O.).

KLEINSCHMIDT (M.). Versuch einer allgemeinen Theorie des Grenzverfahrens. Mit Beispielen. (Progr.) Jena, 1913. 8vo. 66 pp.

KURTH (F.). Herleitung neuer windschiefer Kegelschnitte durch die Bianchische Transformation B_k. (Diss.) Halle, 1913. 8vo. 81 pp.

NIELSEN (J.). Kurvennetz auf Flächen. (Diss.) Kiel, 1913. 8vo. 58 pp.

NITSCHKE (R.). Ueber einige mit den elliptischen Funktionen verwandte Transzendente. (Diss.) Leipzig, 1913. 8vo. 58 pp.

ROUYER (L.). Sur la déformation des quadriques et les surfaces conjugées par rapport à un complexe du second degré. (Thèse.) Toulouse, Privat, 1913. 4to. 127 pp.

SCHALLER (J. G.). Beweis der Richtigkeit des grossen Fermatschen Satzes. Beilage. Grabow i. Meckl., Selbstverlag, 1913. 8vo. 14 pp. M. 1.00

SCHLÖMILCH (O.). See FORT (O.).

VERZEICHNIS über den wissenschaftlichen Nachlass von Sophus Lie. 2te Mitteilung, von A. Guldberg. (Videnskaps selskapets skrifter.) Kristiania, Dybwad, 1913. 8vo. 3+40 pp. M. 1.50

WACKER (H.). Ueber Differentialgleichungen von Fuchsschen Typus mit einem Parameter und ihre Reduzibilität. (Diss.) Strassburg, 1912. 8vo. 96 pp.

II. ELEMENTARY MATHEMATICS.

AMMERMAN (C.). See FORD (W. B.).

ARZELÀ (C.). Trattato di algebra elementare. 3a edizione, 7a impressione. Firenze, Le Monnier, 1913. 16mo. 11+503 pp. L. 3.50

BARDEY (E.). Aufgabensammlung für Arithmetik, Algebra und Analysis. Reformausgabe B. 1ter Teil. 2te, durchgesehene Auflage. Leipzig, Teubner, 1914. 8vo. 6+220 pp. Cloth. M. 2.00

——. Aufgabensammlung für Arithmetik, Algebra und Analysis. Reformausgabe B. 2ter Teil. Leipzig, Teubner, 1914. 8vo. 6+230 pp. Cloth. M. 2.40

BARNARD (S.) and CHILD (J. M.). Exercises from "A New Algebra." Parts 1–4. London, Macmillan, 1913. 8vo. 280 pp. 2s. 6d.

BERGMEISTER (H.) und DUPORT (J. B.). Lehrbuch der Geometrie. Neue Auflage. 1ter und 3ter Teil. Wien, Deuticke, 1913. 8vo. 111 and 94 pp. Cloth. M. 3.80

BERNARDI (G.). Soluzionario di esercizi di trigonometria piana. 3a edizione, nuovamente migliorata. Firenze, Le Monnier, 1913. 16mo. 110 pp. L. 1.25

BERRESHEIN (A.). See HECHT (C.).

BORCHARDT (W. G.) and PERROTT (C. D.). A first numerical trigonometry. With answers. London, Bell, 1913. 8vo. 208 pp. 2s. 6d.

BOS (H.) et REBIÈRE (A.). Eléments de géométrie. 7e édition. Paris, Hachette, 1913. 8vo. 502 pp. Fr. 7.00

BOURLET (C.). Eléments de géométrie. 4e édition revue. Paris, Hachette, 1913. 16mo. 6+383 pp. Cartonné. Fr. 2.50

BUSSLER (F.). Elemente der Mathematik für Realschulen. 2te Auflage. Dresden, Ehlermann, 1913. 8vo. 14+233 pp. Cloth. M. 2.60

CAMMAN (P.) et KEBOUIS (A. G.). Cours élémentaire de géométrie plane. 1er cycle A. 3e édition revue et corrigée. Paris, Gigord, 1914. 12mo. 307 pp. Fr. 1.75

CHILD (J. M.). See BARNARD (S.).

CRACKNELL (A. G.). Junior geometry. With answers. London, Clive, 1913. 8vo. 284 pp. 2s. 6d.

CURSO de algebra elementar. Por R. T. D. 3a edição. Rio de Janeiro, Alves, 1914. 16mo. 271 pp.

DECKERT (A.). Algebra und Planimetrie. Breslau, Goerlich, 1913. 8vo. 3+208 pp. M. 3.30

DELTOUR (P.). Eléments de géométrie. Namur, Wesmael-Charlier, 1913. 98 pp. Cloth. Fr. 2.00

DUPORT (I. B.). See BERGMEISTER (H.).

DUX (O.). Sphärische Trigonometrie und ihre Anwendungen. (Mentor-Repetitorien. 47ter Band.) Berlin-Schöneberg, Mentor, 1913. 66 pp. M. 1.00

EDENHOFER (H.). Lösungen der Diplomvorprüfungs-Aufgaben aus allen Fächern der höheren Mathematik. München, Selbstverlag, 1913. 8vo. 3+141 pp. M. 8.25

FELDMAN (D. D.). See HART (C. A.).

FORD (W. B.) and AMMERMAN (C.). Plane and solid geometry. Edited by E. R. Hedrick. New York, Macmillan, 1913. 12mo. 9+321+33 pp. Cloth. $1.25

——. Solid geometry. New York, Macmillan, 1913. 12mo. Pp. 215–321.

FRIEMEL (R.). Divisions-Tafeln. 2te Auflage. Quedlinburg, Schwanecke, 1913. M. 4.00

GIUDICE (F.). Geometria piana. 3a edizione. Pavia, Mattei, 1914. 8vo. 270 pp. L. 3.00

HART (C. A.) and FELDMAN (D. D.). Key to plane and solid geometry. New York, American Book Co., 1913. 12mo. 196 pp. Cloth. $1.00

HECHT (C.) und BERRESHEIN (A.). Mathematisches Lehrbuch für den Unterricht an Lehrerinnenseminaren. Bielefeld, Velhagen und Klasing, 1914. 8vo. 7+292 pp. Cloth. M. 2.80

HOPKINS (L. A.). See ZIWET (A.).

HYLGERS (T.). Une découverte en calcul mental. Liége, Bovy, 1913. 8 pp. Fr. 1.00

LEHR- und Uebungsbuch für den mathematischen Unterricht. Herausgegeben vom Lehrerverein Hannover-Linden. 1ter Teil: Arithmetik und Algebra. 2ter Teil: Raumlehre. Hannover, Hahn, 1913. 8vo. 204 and 240 pp. M. 3.80

LICHTBLAU (W.) und WIESE (B.). Raumlehre. 1ter Band: Planimetrie. 144 pp. M. 1.75.—3ter Band: Trigonometrie. 183 pp. M. 2.30.—4ter Band: Stereometrie. 200 pp. M. 2.40. Breslau, Hirt, 1913. 8vo.

LIPNOWSKI (A.). Planimetrie. Strelitz, Hittenkofer, 1913. 8vo. 56 pp. M. 2.00

LOGARITHMENTAFEL, 7- und 11stellige. Für die Westentasche. 2te Auflage. Bonn, Huthmacher, 1913. 16mo. 35 pp. M. 0.25

MARTINI (Z. A.). Algebra complementare. 2a edizione, riveduta e aumentata. Livorno, Giusti, 1914. 16mo. 12+143 pp. L. 1.00

NELL (A. M.). Fünfstellige Logarithmen. Völlige Neubearbeitung von L. Balser. 14te Auflage. Giessen, Roth, 1913. 8vo. 6+86 pp. M. 2.00

PERROTT (C. D.). See BORCHARDT (W. G.).

REBIÈRE (A.). See BOS (H.).

REBOUIS (A. G.). See CAMMAN (P.).

REIDT (F.). Einleitung in die Trigonometrie und Stereometrie. 7te Auflage. Berlin, Grote, 1913. 8vo. 32 pp. M. 0.30

RÜEFLI (J.). Lehrbuch der ebenen Trigonometrie. Anhang. Bern, Francke, 1913. 8vo. 20 pp. M. 0.80

SCHULTZE (A.) and SEVENOAK (F. L.). Plane geometry. Revised by A. Schultze. New York, Macmillan, 1913. 12mo. 10+304 pp. $0.80

STAMPER (A. W.). A text-book on the teaching of arithmetic. New York, American Book Co., 1913. 12mo. 284 pp. $1.00

TAVOLE logarithmiche a cinque cifre decimali. Genova, Sordomuti, 1913. 8vo. 27+524 pp.

VOLLKOMMER (M.). Algebra. Klasse 4–6. Nürnberg, Korn, 1913. 8vo. 5+227 pp. Cloth. M. 2.60

WALKER (W. J.). Examples and test papers in algebra. Part 1. London, Mills & Boon, 1913. 8vo. 1s. 6d.

WIESE (B.). See LICHTBLAU (W.).

ZIWET (A.) and HOPKINS (L. A.). Analytic geometry and principles of algebra. Edited by E. R. Hedrick. New York, Macmillan, 1913. 12mo. 8+369 pp. Cloth. $1.60

III. APPLIED MATHEMATICS.

BARBILLION (L.). Transformation des courants, permutatrices, etc. Paris, Geisler, 1913. 8vo. 107 pp.

BAUDRAN (F.) et GUILLON (E.). Moteurs électriques à courant continu. Paris, Geisler, 1913. 8vo. 168 pp.

BAXENDALL (G. A.). See HARRISON (J.).

BENITT (R.). Ueber die Lage des aerodynamischen Druckmittelpunktes an ebenen und gewölbten Flächen. (Diss.) Kiel, 1912. 8vo. 45 pp.

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BERNAYS (P.). Ueber die Bedenklichkeiten der neueren Relativitätstheorie. (*Abhandlungen der Fries'schen Schule.*) Göttingen, Vandenhoeck & Ruprecht, 1913. 8vo. M. 0.80

BETHMANN (H.). Die Kalkulation im Maschinenbau. Neudruck. (Sammlung Göschen. Neue Auflage. Nr. 486.) Berlin, Göschen, 1913. 8vo. 85 pp. Cloth. M. 0.90

BOUASSE (H.). Cours de thermodynamique. . 2e édition, complètement transformée et considérablement augmentée. 2e partie: Machines thermiques; chimie physique. (Tome 2me du Cours de physique.) Paris, Delagrave, 1913. 8vo. 402 pp. Fr. 16.00

BOULVIN (J.). Cours de mécanique, appliquée aux machines. 3e fascicule: Théorie des machines thermiques. 3e édition. Paris, Geisler, 1914. 8vo. 10+554 pp. Fr. 15.00

CAMMAN (P.) et WARISSE (G.). Géométrie descriptive. Classe de 1re C et D. Paris, Gigord, 1914. 8vo. 7+272 pp.

CAPSTICK (J. W.). Sound; an elementary text-book. Cambridge, University Press, 1913. 8vo. 304 pp. 4s. 6d.

CORDEIRO (F. J. B.). The gyroscope. London, Spon, 1913. 8vo. 7+105 pp. 6s. 6d.

CRAIG (R. J.). See NORRIS (E. B.).

DANZER (O.). Die darstellende Geometrie am Realgymnasium. (Progr.) Wien, 1912. 8vo. 63 pp.

DAUGHERTY (R. L.). Hydraulic turbines; with a chapter on centrifugal pumps. New York, McGraw-Hill, 1913. 8vo. 160 pp. Cloth. $2.00

DENIZOT (A.). Das Foucaultsche Pendel und die Theorie der relativen Bewegung. Leipzig, Teubner, 1913. 8vo. 4+76 pp. M. 3.00

DOCK (H.). Photogrammetrie und Stereophotogrammetrie. (Sammlung Göschen. Nr. 699.) Berlin, Göschen, 1913. 8vo. 130 pp. Cloth. M. 0.90

DUNCAN (J.). Applied mechanics for engineers. London, Macmillan, 1913. 8vo. 732 pp. 8s. 6d.

DURHAM (E. B.). Mine surveying. New York, McGraw-Hill, 1913. 12mo. 7+391 pp. Cloth. $3.50

EIFFEL (G.). The resistance of the air and aviation. 2d edition, revised and enlarged. Translated by J. C. Hunsaker. London, Constable, 1913. 4to. 258 pp. Cloth. 42s.

ENSMENGER (K.). Die absolute Bewegung der Himmelskörper. Frankfurt, Kullmann, 1913. 8vo. 27 pp. M. 0.80

FINZEL (A.). Ueber die Reflexion scherender Deformationen an der Grenzfläche zweier Flüssigkeiten. (Diss.) Erlangen, 1913. 8vo. 62 pp.

FLEMING (A. R. M.) and JOHNSON (R.). Insulation and design of electrical windings. London, Longmans, 1913. 8vo. 232 pp. 7s. 6d.

FOWLER. Mechanical engineer's pocket book. London, Scientific Publishing Co., 1914. 12mo. 1s. 6d.

FROMMEL (W.). Radioaktivität. 2te Auflage. Neudruck. (Sammlung Göschen. Neue Auflage. Nr. 317.) Berlin, Göschen, 1913. 8vo. 115 pp. Cloth. M. 0.90

FURTWÄNGLER (P.) und RUHM (G.). Die mathematische Ausbildung der deutschen Landmesser. (Abhandlungen. 4ter Band, 8tes Heft.) Leipzig, Teubner, 1914. 8vo. 6+50 pp. M. 1.60

GIBSON (A. H.). Natural sources of energy. (Cambridge manuals.) Cambridge, University Press, 1913. 16mo. 140 pp. 1s.

GRAETZ (L.). See HANDBUCH.

GUILLON (E.). See BAUDRAN (E.).

HANDBUCH der Elektrizität und Magnetismus. In 5 Bände. Herausgegeben von L. Graetz. 1ter Band, 2te Lieferung. Leipzig, Barth, 1914. 8vo. Pp. 157–420. M. 10.00

HARRISON (J.) and BAXENDALL (G. A.). Practical geometry and graphics for advanced students. Enlarged edition. New York, Macmillan, 1913. 12mo. 14+677 pp. $1.50

HEGEMANN (E.). Lehrbuch der Landesvermessung. 2ter Teil. Berlin, Parey, 1913. 8vo. 8+306 pp. Cloth. M. 13.00

HENKEL (O.). Graphische Statik. 2ter Teil. (Sammlung Göschen. Nr. 695.) Berlin, Göschen, 1913. 8vo. 170 pp. Cloth. M. 0.90

HENSCHKE (E.). Ueber eine Form des Prinzips der kleinsten Wirkung in der Elektrodynamik des Relativitätsprinzips. (Diss.) Berlin, 1912. 8vo. 87 pp.

HÜBNER (E.). Beitrag zur Theorie der isostatischen Reduktion der Schwerebeschleunigung. (Diss.) Berlin, 1913. 8vo. 51 pp.

HUNSAKER (J. C.). See EIFFEL (G.).

INCHLEY (W.). The theory of heat engines. London, Longmans, 1913. 8vo. 502 pp. 7s. 6d.

JOHNSON (R.). See FLEMING (A. P. M.).

KEMPF (P.). See NEWCOMB.

KING (W. I.). Exercises in statistical method. Madison, Wis., University Co-operative Association, 1913. 8vo. 49 pp. $0.40

LAVERGNE (G.). Les turbines. 4e édition. Paris, Gauthier-Villars, 1913. 8vo. 216 pp. Cartonné. Fr. 3.00

LIERNOIT (F. A.). La mathématique scientifique des courses. L'art de miser avec barèmes et progressions. Paris, Drubay, 1913. 16mo. 5+91 pp. Fr. 5.00

LUTZE (G.). Untersuchungen an langsamen elektrischen Schwingungen. (Diss.) Halle, 1913. 4to. 33 pp.

MALISSART (C.). Notions pratiques et indispensables de thermodynamique. Le Roeulx, Bouchat-Simon, 1912. 56 pp. Fr. 3.00

MARÉCHAL (M.). Cours de projections orthogonales. Dison-Verviers, Brach-Juncker, 1910. 40 pp. Fr. 1.00

MISES (R. v.). Elemente der technischen Hydromechanik. 1ter Teil. (Jahnke's Sammlung. Nr. 17.) Leipzig, Teubner, 1914. 8vo. 8+212 pp. Cloth. M. 6.00

MORITZ (F.). Les moteurs thermiques dans leurs rapports avec la thermodynamique. Paris, Gauthier-Villars, 1913. 8vo. 6+297 pp. Fr. 13.00

NERNST (W.) und SCHOENFLIES (A.). Einführung in die mathematische Behandlung der Naturwissenschaften. Kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie. 7te, vermehrte und verbesserte Auflage. München, Oldenbourg, 1913. 8vo. 12+444 pp. Cloth. M. 10.00

NEWCOMB und ENGELMANN. Populäre Astronomie. 5te Auflage. Herausgegeben von P. Kempf. Leipzig, Engelmann, 1914. 8vo. 12+835 pp. Cloth. M. 15.60

NORRIS (E. B.) and CRAIG (R. J.). Shop mathematics. Part 2. Prepared in the extension division of the University of Wisconsin. New York, McGraw-Hill, 1913. 8vo. 226 pp. Cloth. $1.50

PERSONS (W. M.). See ZIZEK (F.).

POPOFF (K.). Sur le mouvement de (108) Hécube. (Thèse.) Paris, Gauthier-Villars, 1913. 4to. 58 pp. Fr. 3.50

RANDALL (J. A.). Exercises in heat. (Wiley technical series.) New York, Wiley, 1913. 4to. Paper. $0.26

RANELLETTI (C.). Elementi di geometria descrittiva. (Manuali Hoepli.) Milano, Hoepli, 1913. 16mo. 12+197 pp. L. 2.00

RASCH (E.). Electric arc phenomena. Translated from the German by K. Tornberg. New York, VanNostrand, 1913. 8vo. 16+194 pp.

RAYMOND (W. G.). The elements of railroad engineering. New edition, revised. New York, Wiley, 1913. 8vo. 16+408 pp. Cloth. $3.50

RUHM (G.). See FURTWÄNGLER (P.).

SCHOENFLIES (A.). See NERNST (W.).

SCHOTT (S.). Statistik. (Sammlung "Aus Natur und Geisteswelt," Band 442.) Leipzig, Teubner, 1913. 12mo. 130 pp. M. 1.25

SIMON (K.). Ratenberechnungstafeln. Karlsruhe, Braun, 1913. 131 pp. Cloth. M. 4.80

SOUTHALL (J. P. C.). The principles and methods of geometrical optics; especially as applied to the theory of optical instruments. New edition. New York, Macmillan, 1913. 8vo. 23+626 pp. $5.50

TORNBERG (K.). See RASCH (E.).

VITALE (G.). La divisione degli archi e degli angoli nel disegno geometrico. Brescia, Unione, 1913. 8vo. 11 pp.

VONDERLINN (J.). Schattenkonstruktionen. Durchgesehene Neudruck. (Sammlung Göschen. Neue Auflage. Nr. 236.) Berlin, Göschen, 1913. 8vo. 118 pp. M. 0.90

WARISSE (G.). See CAMMAN (P.).

WELLSTEIN (J.). Zur theorie der Reibung starker Körper. (Diss.) Strassburg, 1912. 8vo. 30 pp.

WIEN (W.). Vorlesungen über neuere Probleme der theoretischen Physik. Gehalten an der Columbia-Universität in New York im April 1913. Leipzig, Teubner, 1913. 8vo. 4+76 pp. M. 2.40

ZIEEK (F.). Statistical averages; a methodological study. Authorized translation with additional notes and references by W. M. Persons. New York, Holt, 1913. 12mo. 9+392 pp. Cloth. $3.50